# 10. Parton shower and Monte Carlo Methods

In our discussion of jet-production, we have seen that the complexity of perturbative calculations rapidly explodes as the number of external particles increases. Also, our ability to calculate higher-order corrections is quite limited. Only this year, the first NLO results for 2 → 4 processes were obtained and at NNLO only inclusive hadron collider cross sections (i. Drell–Yan, Higgs production) are known.

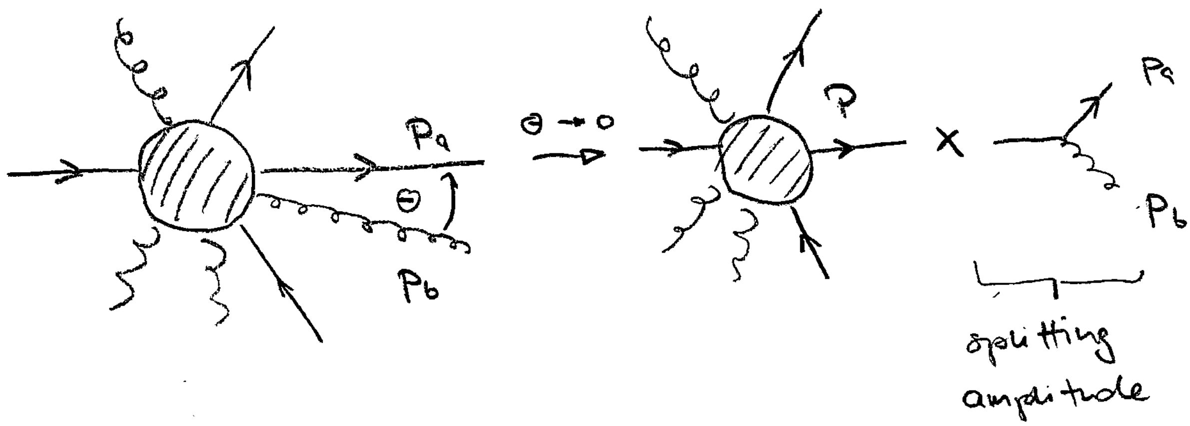
There are two limits, where higher order emissions take a simple form: the emissions factorize in the collinear or soft limit. In the collinear limit one finds

$$d\sigma_{n+1} = d\sigma_n \cdot \underbrace{dP}_{\text{splitting function}}$$

$$P = p_n + p_{n+1}$$

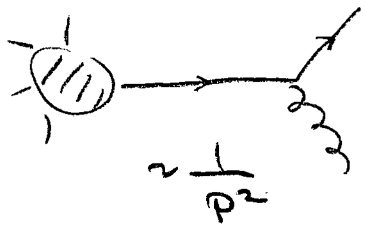
$$d\sigma_{n+1}(p_1, p_2, \dots, p_n, p_{n+1}) = d\sigma_n(p_1, p_2, \dots, p_{n-1}, P) \cdot dP(p_n, p_{n+1})$$

or graphically

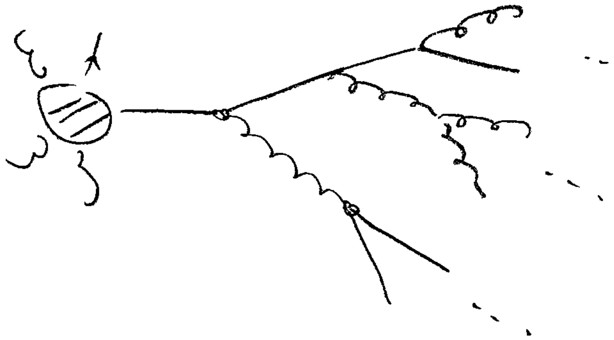



By iterating this relation, one can generate collinear emissions of arbitrary high order.

At the same time, these emissions are enhanced because the propagator denominator $p^2 = (p_a + p_b)^2 \to 0$ as $\theta \to 0$.




A parton shower generates these collinear emissions via a Monte-Carlo process.



Such Monte-Carlo (MC) programs, or event generators, are widely used to analyze collider processes.

We will first derive the splitting functions and then show how the factorization can be used to build a MC event generator.

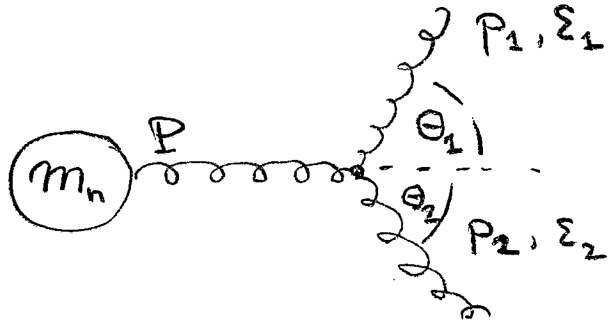
## 10.1. Parton branching and splitting functions

Let's look at the $g \to gg$ splitting first

$P = (E, 0, 0, p)$

$p_1 = (E_1, \sin\theta_1 E_1, 0, \cos\theta_2 E_2)$

$p_2 = (E_2, -\sin\theta_2 E_2, 0, \cos\theta_2 E_2)$



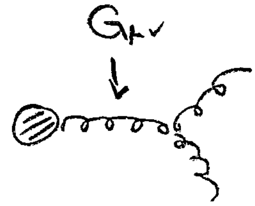

Write $E = z E$

$E = (1-z) E$

$$P^2 = (p_1+p_2)^2 = 2E_1E_2(1-\cos\theta) \cong z(1-z)\theta^2 E^2$$

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \Rightarrow z\theta_1 \cong (1-z)\theta_2$$

$$\theta = \theta_1 + \theta_2 = \theta_1 + \frac{z}{1-z}\theta_2 = \frac{\theta_1}{1-z} = \frac{\theta_2}{z}$$

Internally we have a gluon propagator 


$$G_{\mu\nu}(p) = -\frac{i g_{\mu\nu}}{p^2} + \xi \frac{i\, p_\mu p_\nu}{p^4}$$

$$\begin{aligned} g_{\mu\nu} &= \sum_{\lambda=0}^{3} \varepsilon_\mu(\lambda)\, \varepsilon_\nu^*(\lambda) \\ &= \varepsilon_+^\mu \varepsilon_-^{*\nu} + \varepsilon_-^\mu \varepsilon_+^{*\nu} - \sum_{i=1}^{2} \varepsilon_T^\mu \varepsilon_T^{*\nu} \end{aligned}$$

$$\left.\begin{aligned} \varepsilon_+^\mu &\propto n^\mu \sim p^\mu \\ \varepsilon_-^\mu &\propto \bar{n}^\mu \end{aligned}\right\} \text{ are the unphysical polarizations.}$$

The amputated Green's functions fulfill $p_\mu \Gamma^{\mu\nu\ldots} = 0$ (Ward identity) so that only the physical polarizations give a nonzero contributions, so that we can replace

$$G_{\mu\nu}(p) \longrightarrow \frac{i \sum_{\lambda=1}^{2} \varepsilon_T^\mu(\lambda)\, \varepsilon_T^{*\nu}(\lambda)}{p^2}$$

for the internal propagator.

Let us now evaluate the splitting amplitude for the case, where the internal polarization vector is $\varepsilon_P$.

$p_1 + p_2 = P$

$$V_{ggg} = g_s f_{abc}\, \varepsilon_P^{\mu} \varepsilon_1^{\nu} \varepsilon_2^{\rho}$$

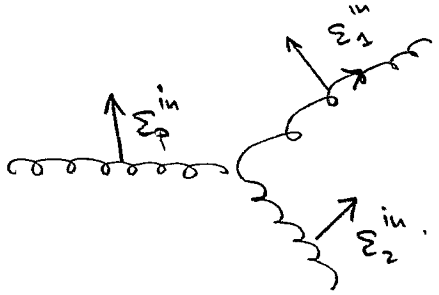
$$\times \Big( g_{\mu\nu} \big(\overset{2p_1^{\rho} + p_2^{\rho}}{P + p_1}\big)_{\rho} + g_{\nu\rho} \big(\overset{-P^{\mu} + 2p_2^{\mu}}{-p_1 + p_2}\big)_{\mu} + g_{\rho\mu} \big(\overset{-2p_2 - p_1}{-p_2 - P}\big)_{\nu} \Big)$$

$$= g_s f_{abc} \Big[ \varepsilon_P \cdot \varepsilon_1\, 2 p_1 \cdot \varepsilon_2 + \varepsilon_1 \cdot \varepsilon_2\, 2 \varepsilon_P \cdot p_2$$

$$\varepsilon_P \cdot \varepsilon_2 ( -2 p_2 \cdot \varepsilon_1 ) \Big]$$

we have used $P = p_1 + p_2$ ; $\varepsilon_i \cdot p_i = 0$.

To evaluate the expression further, let's choose a basis of polarization vectors. It is most natural to choose one of the polarization vectors $\varepsilon_i^{in}$ to lie in the plane of the reaction and the other one out of the plane:




The other one, $\varepsilon_i^{out}$ is chosen perpendicular to the plane.

Explicitly:

$$\varepsilon_p^{out} = \varepsilon_1^{out} = \varepsilon_2^{out} = (0,0,1,0)$$

$$\varepsilon_p^{in} = (0,1,0,0)$$

$$\varepsilon_1^{in} = (0, \cos\theta_1, 0, -\sin\theta_1) \cong (0,1,0,-\theta(1-z))$$

$$\varepsilon_2^{in} = (0, \cos\theta_2, 0, \sin\theta_2) \cong (0,1,0,\theta z)$$

Choosing all polarizations in the plane, one obtains

$$V_{ggg} = 2g_s f_{abc}\Big[-Ez\theta + E(1-z)z\theta - E(1-z)\theta\Big]$$

$$= -2g_s f_{abc} E\theta(1-z+z^2)$$

We need to multiply by $\frac{i}{p^2}$ from the propagator denominator and square

$$\frac{1}{N^2-1}\sum_{\text{colors}}\left|\frac{1}{p^2}V_{ggg}\right|^2 = \frac{\overbrace{f_{abc}f_{abc}}^{C_A(N^2-1)}}{N^2-1}\,\frac{4g_s^2}{p^2}\,\underbrace{\frac{E^2\theta^2}{p^2}}_{\frac{1}{z(1-z)}}(1-z+z^2)^2$$

$$= \frac{4C_A g_s^2}{p^2}\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right] = \frac{4C_A g_s^2}{p^2}F(in,in,in)$$

Repeating the calculations for the other polarizations, we have

| P | 1 | 2 | F |
|---|---|---|---|
| in | in | in | $\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)$ |
| in | out | out | $z(1-z)$ |
| out | in | out | $(1-z)/z$ |
| out | out | in | $z/(1-z)$ |
| all | other combinations | | 0 |

Averaging over incoming and summing over outgoing polarizations, one finds

$$C_A \langle F \rangle = \hat{P}_{gg} = C_A \left[ \frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right]$$

The same splitting function governs the evolution of the gluon PDF.

The singularities at $z=0$ and $z=1$ correspond to soft emissions. In this case one of the gluons carries almost no energy. The singularities only arise for soft gluons with polarizations in the plane.

Instead of $\varepsilon_p^{in}$ and $\varepsilon_p^{out}$, let us consider a polarization vector $\varepsilon_\phi = \cos\phi \, \varepsilon_p^{in} + \sin\phi \, \varepsilon_p^{out}$. In this case one gets

$$F_\phi = \underbrace{\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)}_{\text{unpolarized result.}} + z(1-z)\cos 2\phi$$

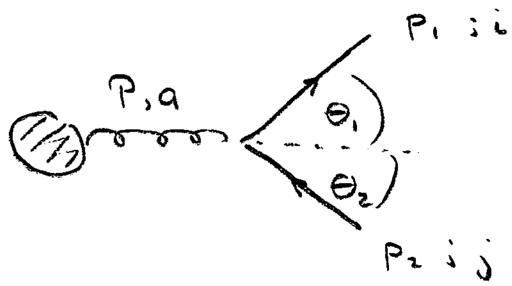
So the branching is enhanced in the plane of the polarization vector, but the dependence is weak:

$$F_\phi = 2{,}25 + 0{,}25 \cos 2\phi \quad \text{for } z = \tfrac{1}{2} .$$

In addition to the $g \to gg$ branching, one also needs the $g \to q\bar{q}$ and the $q \to qg$ branchings.

For $g \to q\bar{q}$, one has

$$V_{q\bar{q}g} = ig(t^a)_{ij}\, \bar{u}(p_1)\gamma_\mu v(p_2)\, \varepsilon_p^\mu$$



To evaluate the expression, one needs to use explicit forms of the spinors, expanded for small angles. In the Dirac basis

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \; ; \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

the spinors are

$$\frac{u_+(p_1)}{\sqrt{E_1}} = \begin{pmatrix} 1 \\ \theta_1/2 \\ 1 \\ \theta_1/2 \end{pmatrix} \qquad \frac{u_-(p_1)}{\sqrt{E_1}} = \begin{pmatrix} \theta_1/2 \\ -1 \\ \theta_1/2 \\ -1 \end{pmatrix}$$

$$\frac{v_+(p_2)}{\sqrt{E_2}} = i \begin{pmatrix} -\theta_2/2 \\ -1 \\ \theta_2/2 \\ 1 \end{pmatrix} \qquad \frac{v_-(p_2)}{\sqrt{E_2}} = i \begin{pmatrix} -1 \\ \theta_2/2 \\ -1 \\ \theta_2/2 \end{pmatrix}$$

After plugging this into the matrix element, one has

$$|\mathcal{M}_{n+1}|^2 = \frac{2g_s^2}{p^2} \overset{\frac{1}{2}}{\overset{\shortparallel}{T_F}} \mp |\mathcal{M}_n|^2$$

| P | 1 | 2 | $\mp$ |
|---|---|---|---|
| in | $\pm$ | $\mp$ | $\frac{1}{2}(1-2z)^2$ |
| out | $\pm$ | $\mp$ | $1/2$ |

Note that there are no ~~soft~~ singularities ($z \to 0$, or $z \to 1$).

They only appear in gluon emissions.

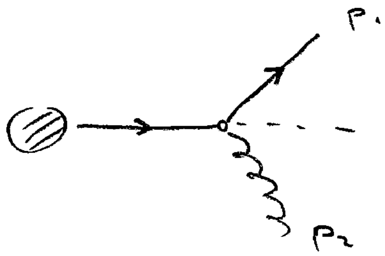
Averaged splitting function:

$$\hat{P}_{qg}(z) = T_F \langle F \rangle = T_F\left[z^2 + (1-z^2)\right]$$

$$F_\phi = z^2 + (1-z^2) - 2z(1-z)\cos(2\phi)$$

Finally, for $q \to qg$, one obtains

$$|M_{n+1}|^2 = \frac{2g_s^2}{p^2} C_F \, F \, |M_n|^2$$




| P | 1 | 2 | |
|---|---|---|---|
| $\pm$ | $\pm$ | in | $\frac{1}{2}(1+z)^2/(1-z)$ |
| $\pm$ | $\pm$ | out | $\frac{1}{2}(1-z)$ |

Averaged splitting function

$$\hat{P}_{qq}(z) = C_F \langle F \rangle = C_F \frac{1+z^2}{1-z}$$

$$F_\phi = \frac{1+z^2}{1-z} + \frac{2z}{1-z}\cos 2\phi$$

We have derived results for the $n+1$ particle amplitude squared $|\mathcal{M}_{n+1}|^2$ in terms of $|\mathcal{M}_n|^2$.

To get a relation for the cross sections, we also need to relate the phase-space in the two cases.

To do so, remember that

$$\int d^4p\, \Theta(p^0)\, \delta(p^2 - M^2) = \int \frac{d^3p}{2E}$$

therefore

$$\int dM^2 \int d^4p\, \Theta(p^0)\, \delta(p^2 - M^2)$$

$$= \underline{\int d^4p\, \Theta(p^0) = \int dM^2 \int \frac{d^3p}{2E}}$$

Now rewrite the $(n+1)$-particle phase-space

$$\int d\Phi_{n+1} = \int d\Phi_{n-1} \int \frac{d^3p_1}{2E_1(2\pi)^3} \int \frac{d^3p_2}{2E_2(2\pi)^3}$$

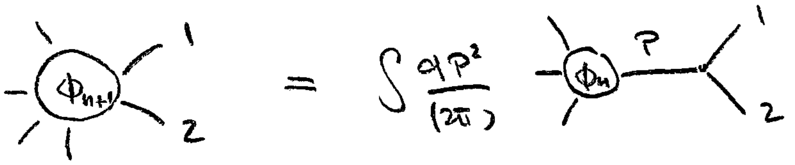
$$= \int d\Phi_{n-1} \int d^4P\, \Theta(P^0) \int \frac{d^3p_1}{2E_1(2\pi)^3} \int \frac{d^3p_2}{2E_2(2\pi)^3}\, \delta^{(4)}(P - p_1 - p_2)$$

M2

$$= \int \frac{dP^2}{(2\pi)} \int d\Phi_{n-1} \int \frac{d^3P}{2E\,(2\pi)^3} \int \frac{d^3p_1}{2E_1(2\pi)^3} \int \frac{d^3p_2}{2E_2\,(2\pi)^3}\,(2\pi)^4\,\delta(P-p_1-p_2)$$

$$= \int \frac{dP^2}{(2\pi)} \int d\underline{\underline{\Phi_n}} \cdot \int \frac{d^3p_1}{2E_1(2\pi)^3} \int \frac{d^3p_2}{2E_2\,(2\pi)^3}\,(2\pi)^4\,\delta(P-p_1-p_2)$$

Pictorially:




Now we want to approximate the integration over the two-particle phase-space in the limit where $\theta \to 0$.

$$\int dP^2 \int \frac{d^3p_1}{2E_1} \int \frac{d^3p_2}{2E_2}\,\delta^4(P-p_1-p_2)$$

$$= \int dP^2 \int \frac{d^3p_1}{2E_1} \int d^4p_2\,\Theta(p_2^0)\,\delta(p_2^2)\,\delta^{(4)}(P-p_1-p_2)$$

$$= \int dP^2 \int \frac{d^3p_1}{2E_1}\,\Theta(E-E_2)\,\delta((P-p_1)^2)$$

$$= \int dP^2 \int dE_1\,\frac{E_1}{2} \int \cos\theta_1 \int d\varphi\;\Theta(E-E_2)\,\delta\left[P^2-2EE_2(1-\cos\theta_1)\right]$$

$\int d\theta \sin\theta \approx \frac{1}{2}\int d\theta^2$ $\qquad 1-\cos\theta_1 \approx \frac{\theta_1^2}{2}$

$$= \int dp^2 \int_0^1 dz \, \frac{zE^2}{2} \, \frac{1}{2}\int d\theta_1^2 \int d\varphi \, \delta\left[p^2 - E^2 z \, \theta_1^2\right]$$

$$= \int dp^2 \int_0^1 dz \, \frac{1}{4} \int d\varphi$$

So we have

$$\int d\Phi_{n+1} = \frac{1}{4(2\pi)^3} \int_0^\infty dp^2 \int_0^1 dz \int_0^{2\pi} d\varphi \int d\Phi_n$$

For the differential cross section, we get

$$d\sigma_{n+1} = d\sigma_n \, \frac{dp^2}{p^2} \, \frac{d\varphi}{(2\pi)} \, \frac{\alpha}{2\pi} \, C \cdot F \, dz$$

[Note: for $g \to gg$, the $d\sigma_{n+1}$ contains a factor $\frac{1}{2}$ for identical particles.]

If we don't care about the $\varphi$-dependence, we can average

$$d\sigma_{n+1} = d\sigma_n \, \frac{dp^2}{p^2} \, \frac{\alpha}{2\pi} \, \hat{P}(z) \, dz$$