COLLIDER PHYSICS & QCD

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Introduction

We are all eagerly awaiting the start of the Large Hadron Collider at CERN. The (re)start should happen towards the end of the year and the machine will initially run at 3,5 TeV per beam. The design energy of 7 TeV / beam should be reached later.

The LHC will be able to test a wide variety of new physics models. It will find the Higgs boson, or whatever else is responsible for electroweak symmetry breaking.

However, to achieve the physics goals of the LHC program will require a great deal of understanding of standard Model physics. The biggest challenge in this respect are QCD effects.

QCD at high energy colliders will be the focus of these lectures. In the first part we'll provide an introduction to perturbative QCD. After this, we'll try to answer the question why (or better when) the use of perturbation theory is appropriate. The naive answer to this question is that QCD perturbation theory can be used in high-energy collisions because of asymptotic freedom, i.e. the statement that the strong interaction becomes weak at high energies. However, even in the highest-energy colliders, we observe hadrons, and not quarks and gluons – so how can perturbation theory ever be appropriate?

Indeed, most observables at colliders cannot be calculated perturbatively. However, for a limited set of observables the non-perturbative hadronic effects are suppressed like $m_{\text{Hadron}}/E_{\text{c.m.}}$ and are thus small. At hadron colliders, no such observables exist, because all reactions depend on non-perturbative properties of the incoming hadrons. However, this dynamics is the same irrespective of the final state of the collision. Using factorization theorems, it can be seperated from the rest of the scattering process.

Because of the complexity of hadron collisions, we'll first discuss e^+e^- collisions and move then to e^-p and then pp scattering.

Since the classical factorization "proofs" are very technical, many books avoid discussing the issue in any detail. However, in the past few years, an effective field theory has been developed which provides a better language to address these issues. The framework is called Soft-Collinear Effective Theory and will be covered as part of the course.

Literature:

QCD

There are many excellent books covering QCD.

Two I particularly like are

- * Peskin & Schröder "Intro to QFT"
- * G. Sterman "An intro to QFT"

Sterman is the expert on factorization and his book is one of the few to address the topic.

Collider physics

The most useful reference for our course is

- * Ellis, Stirling and Webber "QCD and Collider Physics"

another relevant book is

- * Barger and Phillips "Collider physics"

however this one is much more focussed on phenomenology. (QCD only appears at p. 200.)

2. Non-abelian gauge theory

QCD is an example of a non-abelian gauge theory. (The electroweak theory is another example!) These theories are generalizations of electrodynamics which is an abelian gauge theory.

2.1. Gauge invariance

Consider complex field $\psi(x)$ and perform phase redefinitions

$$\psi(x) \to e^{i\alpha(x)} \, \psi(x) \qquad \text{"gauge transformation"}$$

We now want to construct a theory which is invariant under this transformation. $\psi^+(x) \cdot \psi(x)$ is invariant.

However, derivatives no longer make sense

$$\underset{\substack{\nearrow \\ \text{unit vector}}}{n^\mu} \partial_\mu \psi(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\psi(x+n\varepsilon) - \psi(x) \right]$$

Different transformation law.

In order to be able to compare fields at different points we need to introduce a quantity which compensates phase difference at neighbouring points, e.g. "Link field"

$$\underline{U(y,x) \Rightarrow} \, e^{i\alpha(y)} \, U(y,x) \, e^{-i\alpha(x)}$$

$$\left[U(x,x) = 1 \qquad U(x,y) = U^{+}(y,x) \qquad |U| = 1 \right]$$

Then $$n^\mu D_\mu \psi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\psi(x+\varepsilon n) - U(x+n\varepsilon, x)\psi(x) \right]$$

Expand $U(x+n\varepsilon, x) = 1 - \varepsilon i e n^\mu A_\mu(x) + O(\varepsilon^2)$.

The field A_μ is called a <u>connection</u> (or gauge field)

Transformation: $A_\mu(x) \to A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x)$

Check:

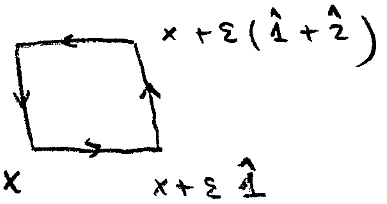
$$D_\mu \psi \underline{\Rightarrow} \{ \partial_\mu + ie (A_\mu - \tfrac{1}{e} \partial_\mu \alpha) \} e^{i\alpha} \psi$$

$$= e^{i\alpha} \{ \partial_\mu + ieA_\mu \} \psi = e^{i\alpha} D_\mu \psi$$

we can now construct a gauge invariant Lagrangian for ψ which includes derivative terms.

In terms of $U(y,x)$ we could consider closed curves

e.g.



To get an invariant using differential quantities consider

$$[D_\mu , D_\nu] \psi(x) \longrightarrow e^{i\alpha} [D_\mu , D_\nu] \psi$$

$$[D_\mu , D_\nu] \psi = ([\partial_\mu , \partial_\nu] + ie \, \overset{(\partial_\mu A_\nu)}{[\partial_\mu , A_\nu]} - [\partial_\nu , A_\mu]$$

$$- e^2 \underbrace{[A_\mu , A_\nu]}_{0}) \psi$$

$$= ie (\partial_\mu A_\nu - \partial_\nu A_\mu) \psi$$

So $[D_\mu, D_\nu] = ie F_{\mu\nu}$

We can now write down the most general renormalizable Lagrangian, i.e. all operators up to dimension 4.

$[\psi] = 3/2$, $[D_\mu] = 1$

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - m \bar{\psi} \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$- ic \underbrace{\varepsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu}}_{\text{violates } P, T}$$

2.2. The Yang-Mills Lagrangian

'54 Yang and Mills proposed to generalize invariance under phase rotation to an arbitrary continuous group (Lie group).

The simplest example is the rotation group $SU(2)$ $(\cong O(3))$. Consider $\psi = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ and demand invariance under

$$\psi(x) \longrightarrow \exp\left(i\,\alpha^i(x)\frac{\sigma^i}{2}\right)\psi(x) = V(x)\,\psi(x)$$

The σ^i are the usual Pauli matrices:

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] \underline{\underline{=}}\ i\varepsilon^{ijk}\frac{\sigma^k}{2} \quad ; \quad \sigma^i = (\sigma^i)^+ \quad tr(\sigma^i) = 0$$

<u>Note</u>: V is unitary $VV^+ = 1$ and $\text{Det}\, V = 1$

When discussing spin, the Pauli matrices describe rotations in real space. In contrast, we are considering rotations in field space, i.e. rotations of the fermions ψ_1 & ψ_2 into each other.

We now proceed in exactly the same way as last time. We introduce a link field

$$U(y,x) \underline{\Rightarrow} V(y)\, U(y,x)\, V^{\dagger}(x)$$

$$\left[U(x,x) = 1 ,\quad U^{+}U = 1 ,\quad U(x,y) = U^{+}(y,x) \right]$$

and define the covariant derivative

$$n^{\mu} D_{\mu} \psi(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[\psi(x+n\varepsilon) - U(x+n\varepsilon, x)\, \psi(x) \right]$$

and expand the link field:

$$D_{\mu} = \partial_{\mu} - ig A_{\mu}^{i} \frac{\sigma^{i}}{2}$$

Instead of a single photon A_μ, we get $\underline{\text{three}}$ gauge fields A^i_μ. (In the weak interaction, they correspond to $W^\pm, Z$.)

Let's derive the transformation law for A^i_μ.

$$U(x+n\varepsilon, x) = 1 + i\varepsilon g n^\mu A^i_\mu \frac{\sigma^i}{2}$$

$$\longrightarrow V(x+\varepsilon n)\left[1 + ig\varepsilon n^\mu A^i_\mu \frac{\sigma^i}{2}\right] V^+(x)$$

$$V(x+\varepsilon n)\, V^+(x) = \left[(1+\varepsilon n^\mu \partial_\mu)\, V(x)\right] V^+(x)$$
$$= 1 - \varepsilon n^\mu V(x)\, \partial_\mu V^+(x)$$

So $A^i_\mu \frac{\sigma^i}{2} \longrightarrow V(x)\left[A^\mu_i \frac{\sigma^i}{2} + \frac{i}{g}\partial^\mu\right] V^+(x)$

Complicated! Expand $V(x) = 1 + i\alpha^i \frac{\sigma^i}{2} + O(\alpha^2)$.

$$A^i_\mu \frac{\sigma^i}{2} \longrightarrow A^\mu_i \frac{\sigma^i}{2} + \underline{\frac{1}{g}}\, \partial^\mu \alpha^i \frac{\sigma^i}{2} + i\, \underbrace{\left[\alpha^i \frac{\sigma^i}{2}, A^j_i \frac{\sigma^j}{2}\right]}_{\text{new!}}$$

Exercise: check that $D_\mu \psi \to V D_\mu \psi$.

To obtain the kinetic term for the gauge field, we again consider $[D_\mu, D_\nu]\,\psi(x)$.

As in the abelian case, $[D_\mu, D_\nu]\psi$ does not contain a derivative on ψ.

$$[D_\mu, D_\nu] = i g F^i_{\mu\nu} \frac{\sigma^i}{2}$$

$$F^i_{\mu\nu} \frac{\sigma^i}{2} = \partial_\mu A^i_\nu \frac{\sigma^i}{2} - \partial_\nu A^i_\mu \frac{\sigma^i}{2} - ig \left[A^i_\mu \frac{\sigma^i}{2}, A^j_\nu \frac{\sigma^j}{2} \right]$$

Now use: $\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2} \right] = i \varepsilon^{ijk} \frac{\sigma^k}{2}$

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g \varepsilon^{ijk} A^i_\mu A^k_\nu$$

The transformation law $[D^\mu, D^\nu] \to V [D^\mu, D^\nu] V^+$ implies that $F^i_{\mu\nu}$ is not gauge invariant.

However $\mathcal{L} = -\frac{1}{2} \operatorname{tr}\left[\left(F^i_{\mu\nu} \frac{\sigma^i}{2} \right)^2 \right] = -\frac{1}{4} \underset{\substack{\uparrow \\ \text{sum over } i}}{(F^i_{\mu\nu})^2}$

is gauge inv.

The most general renormalizable $\mathcal{L}$ is

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - m \bar{\psi}\psi - \frac{1}{4} (F^i_{\mu\nu})^2 \underbrace{- i c\, \varepsilon^{\mu\nu\rho\sigma} F^i_{\mu\nu} F^i_{\rho\sigma}}_{\text{violates } P,\, T}$$

Our discussion easily generalizes to an arbitrary continuous symmetry group.

Yang & Mills wanted to explain the strong interaction with an SU(2) gauge field acting on $\psi = \begin{pmatrix} p \\ n \end{pmatrix}$. However the gauge fields are massless, so why were they not observed? (Also, it was not clear at the time how to quantize YM theory.)

2.3 Lie groups, SU(N)

Lie groups represent continuous symmetries, e.g. rotations. To build gauge theories, we are interested in groups with finite-dim unitary representations, so called compact Lie groups.

We are interested in groups which can be generated from repeated action of an infinitesimal group elements

"generators"

$$g(\alpha) = 1 + i\alpha^a t^a + O(\alpha^2)$$

The structure of the group is determined by the commutation relations

$$[t^a, t^b] = i f^{abc} t^c$$

"structure constants"

Note Baker-Campbell Hausdorff

$$e^{i\alpha\cdot t} e^{i\beta\cdot t} = e^{i(\alpha+\beta)\cdot t - \frac{1}{2}[\alpha\cdot t, \beta\cdot t] + \dots}$$

Terminology for Lie groups

Simple: Cannot split generators in two commuting sets

Semi-simple: Does not contain generators which commutes with all others, ie. no U(1) factors.

Jacobi identity

$$[t^a,[t^b,t^c]] + [t^b,[t^c,t^a]] + [t^c,[t^a,t^b]] = 0$$

implies

$$[[T^a, i f^{bcd} T^d]] = i f^{ade} f^{bcd} T^e + \dots$$

$$f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0$$

There exist only a small number of such groups (Killing & Cartan)

1.) SU(N) $\quad \vec{\xi} \to U\vec{\xi} \quad$ (ξ complex)

Complex n-dim unitary matrices $U^+ U = 1$ with $\det(U) = 1$

$N^2 - 1$ generators, which are Hermitian traceless matrices $\quad T^+ = T \quad \mathrm{tr}(T) = 0$

2.) Orthogonal transformations $UU^+ = 1$ of n-dim vectors. Rotation group.

$N(N-1)/2$ generators

3.) Symplectic transformations of N-dim. vectors $Sp(N)$. These trafo's leave the product

$$\vec{y}^T E \vec{x} \quad \text{invariant} \quad E = \begin{pmatrix} 0 & 1\!\!1_{\frac{N}{2}\times\frac{N}{2}} \\ -1\!\!1_{\frac{N}{2}\times\frac{N}{2}} & 0 \end{pmatrix}$$

It has $N(N+1)/2$ generators

In addition, there are five exceptional Lie algebras denoted by

$$G_2, F_4, E_6, E_7, E_8$$

A d-dim representation R is a set of d×d matrices which satisfy the commutation relations.

Choose t_R^a such that

$$\mathrm{tr}\left[t_R^a t_R^b\right] = T_R \ \delta^{ab}$$

For each representation, there is a complex conjugate rep.

$$\phi \longrightarrow (1 + i\alpha^a t_R^a)\phi$$

$$\phi^* \longrightarrow (1 - i\alpha^a (t_R^a)^*)\phi \quad .$$

$$\rightsquigarrow t_{\bar{R}}^a = -(t_R^a)^* = -(t_R)^T .$$

The structure constants f_{abc} form a representation, the so-called adjoint repr.

$$(t_A^c)_{ab} = -i f_{abc}$$

The commutation relations for t_A^c are the Jacobi identity.

The simplest representation for $SU(N)$ is on the space of N-dim vectors and is called the fundamental rep.

We choose t^a, such that

$$(*) \qquad \mathrm{tr}\left[t_F^a t_F^b \right] = T_F \delta^{ab} = \frac{1}{2} \delta^{ab}$$

The Casimir operator $T_R^2 = \sum_a t_R^a t_R^a$ commutes with all group elements. For an irreducible representation, it follows (Schur's Lemma)

$$T_R^2 = C_R \cdot \mathbb{1} .$$

⌈ Irreducible: block diagonalize representation. If there is only 1 block, the rep. is irred.
⌊

Let's calculate C_F for $SU(N)$

$$\mathrm{tr}\left[t_F^a t_F^a \right] = \frac{1}{2} \sum_{a,b} \delta^{ab} = \frac{1}{2} (N^2 - 1)$$

$$= \mathrm{tr}\left[C_F \mathbb{1} \right] = C_F \, N$$

$$\leadsto \quad C_F = \frac{N^2 - 1}{2N}$$

Without proof $C_A = N$.

An explicit basis of generators for SU(3)

are the Gell-Mann matrices

$$t^1 = \frac{\lambda^1}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

... (see any book)

$$t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}.$$

However, in practice one avoids the explicit rep. and tries to express everything in terms of Casimir invariants, or uses the Fierz relation of SU(N)

$$t^a_{ij}\, t^a_{kl} = \frac{1}{2}\left(\delta_{il}\,\delta_{kj} - \frac{1}{N}\,\delta_{ij}\,\delta_{kl} \right)$$

QCD

Up to now, we have discussed a general gauge theory. QCD in particular is an SU(3) gauge theory with six sets of quark fields:

	e_q		
"up-type"	u $\frac{2}{3}$ $m_u \sim 3$ MeV	c $\frac{2}{3}$ $m_c \sim 1{,}2$ GeV	t $\frac{2}{3}$ $m_t \simeq 172$ GeV
"down-type"	d $-\frac{1}{3}$ $m_d \sim 5$ MeV	s $-\frac{1}{3}$ $m_s \simeq 100$ GeV	b $-\frac{1}{3}$ $m_b \simeq 4{,}6$ GeV

Each fermion transforms under the fundamental rep. of color: e.g. $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{array}{l} \text{"red"} \\ \text{"green"} \\ \text{"blue"} \end{array}$

The associated gauge field is called gluon.

The heavier quarks decay because of the weak interaction.

2.4 A brief history of the discovery of QCD

- 50's: "Particle Zoo": a very large number of hadrons are discovered. Attempts to develop a field theory of the strong interaction all fail. Among those failed attempts:

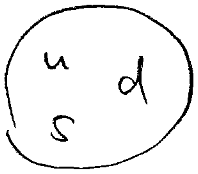
- '54: Yang-Mills propose non-abelian gauge theory (seems to imply new unobserved massless particle)

Dyson '60: "The correct theory will not be found in a hundred years!"

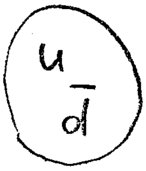
General believe that QFT does not work for strong interactions

- '64 Quark model (Gell-Mann, Zweig)

Hadrons look like they are made from hypothetical quarks



baryons



mesons

'65 Additional quantum number "color"

$$\Delta^{++} \sim | u\uparrow, u\uparrow, u\uparrow \rangle \ ?$$

spin $3/2$

cannot be in the same state (Pauli exclusion)

must be anti symm with respect to exchange of quarks

Work if we introduce three "colors" of each quark flavor and ask that all hadrons are color neutral
(Han, Nambu, Greenberg; Bogolyubov, Struminski, Tavchelidze)

Late 60s $e^- p \to e^- X$ scattering exp's at SLAC

can be explained by assuming that the electrons scatter elastically on free constituents ("partons") of the proton

Feynman: "We shall ... think of the incoming proton as a box of partons sharing the momentum and practically free"

Parton model . parton - parton interactions should become weak at high energies "asymptotic freedom".

'68 Callan & Gross show a way of distinguishing the spin of the partons. Exp strongly favors spin ½.

'71 't Hooft & Veltman show that YM theories are renormalizable

'72 Gell-Mann & Fritzsch propose SU(3) gauge theory with quarks and gluons (at a conference). In '73 paper with Leutwyler they explain the advantages of this model.

'73 Politzer; Gross & Wilczek show that YM theory is asymptotically free!

(Gross and Wilczek set out to proof that <u>no</u> field theory is asymptotically free - YM is the only exception)

3. Perturbative QCD

Now that we have constructed the Lagrangian of non-abelian gauge theory, we want to quantize these theories. By far the simplest method is to use the path integral formulation of QFT.

As in the case of QED the naive path integral over the gauge field

$$Z = \int \mathcal{D}A_\mu \exp\left[iS[A_\mu] \right]$$

is ill-defined. The problem arises from gauge invariance: all gauge configuration related by a gauge transformation have the same weight in the integral. In particular, all configurations where $A_\mu(x) = \frac{i}{g} V(x) \partial_\mu V(x)$; $V(x) = \exp(i\alpha^a t^a)$

have $S[A_\mu]=0$, since they are obtained from $A_\mu=0$ by a gauge transformation. In order to get a meaningful path integral, we want to factor out the integral over the gauge group, since it will not contribute to gauge invariant expectation values.

A method to achieve this was found by Faddeev and Popov in '67. We'll first illustrate it for an ordinary integral before applying it to the path integral over the gauge field.

3.1. The Fadeev-Popov Lagrangian

Let's first consider a two-dimensional integral to illustrate the procedure:

$$I = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \;\; f(x,y)$$

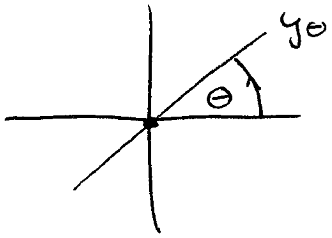
where $f(x,y)$ is rotationinvariant ($\equiv$ gauge invariant in the QCD case). We want to write this as

$$I = \underbrace{\int_0^{2\pi} d\theta}_{\substack{\text{integral} \\ \text{over the symmetry group ("gauge group")}}} \int_0^{\infty} dr \; F(r)$$

How can we achieve this if we don't know $F(r)$?

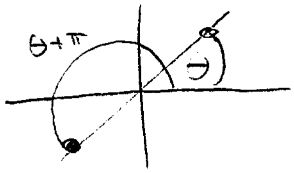
Fix a direction ($\equiv$ gauge) by the condition

$$y_\theta = x \sin\theta + y \cos\theta = 0$$



To be rotation invariant: integrate over all directions:

$$\int_0^{2\pi} d\theta \; \delta(y_\theta) \left|\frac{\partial y_\theta}{\partial \theta}\right| = 2$$



$$\frac{\partial y_\theta}{\partial \theta} = x\cos\theta - y\sin\theta = \sqrt{x^2+y^2}$$

$$\tan\theta = -\frac{y}{x}$$

Now insert this into the integral

$$I = \int d\theta \int dx \int dy \; \delta(y_\theta)\, \frac{1}{2}\sqrt{x^2+y^2}\, f(x,y)$$

and rotate your coordinate system by θ:

$$y' = y_\theta$$
$$x' = x\cos\theta - y\sin\theta$$

$$I = \int_0^{2\pi} d\theta \int dx' \int dy' \; \delta(y')\, \underbrace{\frac{1}{2}\sqrt{x'^2+y'^2}\, f(x',y')}_{\text{by rotation invariance}}$$

$$= \underbrace{2\pi}_{\substack{\text{Volume} \\ \text{of the symmetry group}}} \times \int dx' \int dy' \; \delta(y')\, \frac{1}{2}\sqrt{x'^2+y'^2}\, f(x',y')$$

$$= 2\pi \int dx' \, \frac{x'}{2}\, f(x',0)$$

Now the real thing: consider the path integral

$$Z = \int \mathcal{D}A_\mu \exp\left[i \int d^4x \left(-\frac{1}{4}(F^i_{\mu\nu})^2\right)\right]$$

(the complication only concerns the gauge part, so we don't include fermions for the moment.)

We now want to introduce the analogue of the ray $y_0 = 0$. First choose a gauge fixing condition

$$g(A) = \partial_\mu A^a_\mu = 0 \qquad \text{("Lorentz gauge")}$$

or

$$g(A) = n_\mu \cdot A^{a\mu} = 0 \qquad \text{("axial gauge")}$$

are popular choices.

Equally well, we can impose the condition

$$g(A^\alpha) = \omega(x)$$, where $\omega(x)$ is some arbitrary function of x and

$$(A^\alpha)^a_\mu t^a = V(x)\left[A^a_\mu t^a + \frac{i}{g}\partial_\mu\right]V^\dagger(x)$$

$$V(x) = \exp(i\alpha^a t^a)$$

This is now the analogue of $y_\theta = 0$.

To preserve gauge invariance, we integrate over all $\alpha^a(x)$, in the same way we integrated over all angles

$$\prod_x \delta(g(A(x)) - \omega(x)))$$

$$\int \mathcal{D}\alpha \, \delta(g(A^\alpha) - \omega) \det\left(\frac{\delta g(A^\alpha)}{\delta \alpha} \right) = 1$$

⌈ = 1 if the solution is unique. It is not: there are so called "Gribov-copies", however they do not contribute in PT.
⌊

$$Z = \int \mathcal{D}\alpha \int \mathcal{D}A_\mu \, \delta(g(A^\alpha) - \omega) \det\left(\frac{\delta g(A^\alpha)}{\delta\alpha} \right)$$
$$\exp\left[i S[A] \right]$$

Now change variables

$$t^a A^a_\mu \to t^a A'^a_\mu = t^a (A^\alpha)^a_\mu = V [A^a_\mu t^a + \frac{i}{g}\partial_\mu] V$$

$$Z = \left[\int \mathcal{D}\alpha\right] \int \mathcal{D}A'_\mu \; \delta(g(A') - \omega) \overbrace{\det\left(\frac{\delta g(A')}{\delta \alpha}\right)}^{\alpha\text{-indep!}}$$

$$\exp\left[i S[A'] \right] \Big]$$

So we have factored out the integral over the gauge group. We still want to bring the path integral into managable form.

1.) Note that Z is independent of ω, so let's integrate over it with Gaussian weight

$$Z = \frac{1}{N} \int \mathcal{D}\omega \, e^{-i \int d^4x \frac{\omega^2}{2\xi}} \; Z$$

$$= \frac{1}{N} \left[\int \mathcal{D}\alpha\right] \int \mathcal{D}A_\mu \, \exp\left[i S[A] - i \int d^4x \frac{1}{2\xi} \overset{\overset{\text{e.g. } (\partial_\mu A^\mu)^2}{\downarrow}}{g(A')^2} \right]$$

2.) Let's represent $\det\left(\frac{\delta g(A)}{\delta \alpha}\right)$ as a functional integral.

$$\det\left(\frac{\delta g^a}{\delta\alpha^b}\right) = \int \mathcal{D}\bar{c}\,\mathcal{D}c\,\exp\Big[i\int d^4y\int d^4z$$
$$-\bar{c}^a(y)\left[\delta g^a(y)/\delta\alpha^b(z)\right]c^b(z)\Big]$$

The fields c & $\bar{c}$ are the Feynman-DeWitt-Faddeev-Popov ghosts. They are scalars under the Lorentz group, but anti-commuting!

Let's calculate the functional derivative $\delta g^a(A(y))/\delta\alpha^b(z)$ for Lorentz gauge

$$t^a g^a(A^\alpha) = \partial_\mu A^{\alpha a}_\mu t^a = \partial^\mu V(x)\left[A^a_\mu t^a + \frac{i}{g}\partial_\mu\right]V^\dagger(x)$$

$$= \partial^\mu\Big[A^a_\mu t^a + \frac{1}{g}\partial_\mu\alpha^a t^a + \underbrace{i\left[\alpha^a t^a, A^b_\mu t^b\right]}_{-f^{bca}\alpha^b A^c_\mu t^a}\Big]$$

$$g^a(A^\alpha) = A^a_\mu + \frac{1}{g}\partial_\mu D^\mu\alpha^a$$

$\partial_\mu + i t^a A^a$

$$\frac{\delta g^a(A^\alpha(y))}{\delta\alpha^b(z)} = \frac{1}{g}\partial^\mu\underbrace{D_\mu}_{\delta^{ab}\partial_\mu - g f^{abc}A^c_\mu}\delta(y-z)$$

To summarize

have absorbed $\frac{1}{g}$ into c-field

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \frac{1}{2}\xi(\partial^\mu A^a_\mu)^2 + \bar{c}^a(-\partial^\mu D^{ab}_\mu)c^b$$

$$+ \bar{\Psi}(i\not{D} - m)$$

$$D^{ab}_\mu = \delta^{ab}\partial_\mu - g f^{abc} A^c_\mu$$

Note for QED $D^{ab}_\mu \Rightarrow \partial_\mu$: the ghosts can be integrated out since they don't interact.

We know that the theory is gauge invariant, but it is no longer manifest. However, even the gauge fixed Lagrangian has a symmetry, the so called BRST (Becchi, Rouet, Stora & Tyutin) symmetry (see e.g. Peskin) which can be used to derive Ward identities.

The ghost contributions are unphysical and cancel the equally unphysical contributions from timelike and longitudinal polarizations of the gluon.

In QED, the unphysical polarizations do not appear, as long as all external photons are transversly polarized. However in QCD they appear in loop diagrams. The ghost loops cancel their contribution. (Note that ghost loops involve a factor -1 because the fields anticommute.)

3.2. Feynman rules

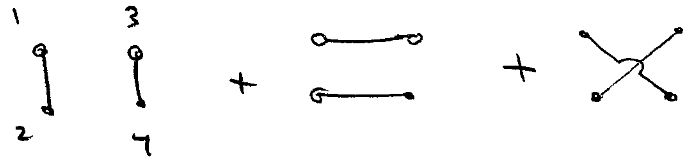
Let's briefly recapitulate how they are derived for a scalar theory (for the purpose of Feynman rules, we can view each component of the gluon field as a scalar field).

E.g.
$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}}_{\mathcal{L}_0} \underbrace{- \frac{\lambda}{4!}\phi^4}_{+\ \mathcal{L}_I}$$

Define
$$\langle \phi_1 \dots \phi_n \rangle = \langle \phi(x_1) \cdots \phi(x_n) \rangle = \int \mathcal{D}\phi\, \phi(x_1) \dots \phi(x_n)\, e^{i\int d^4x\, \mathcal{L}_0}$$

Wick's theorem
$$\langle \phi_1 \dots \phi_n \rangle = \sum_{\substack{\text{all pairings}\\ i_1 \dots i_n}} \underbrace{\overset{x_{i_1} \;\text{———}\; x_{i_2}}{\langle \phi_{i_1} \phi_{i_2} \rangle}}_{\text{free propagator.}} \langle \phi_{i_3} \phi_{i_4} \rangle \dots \langle \phi_{i_{n-1}} \phi_{i_n} \rangle$$

For example

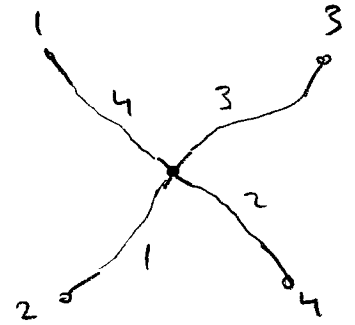
$\langle \phi_1 \dots \phi_4 \rangle =$ 

Interacting theory

$$\langle \phi_1 \cdots\cdots \phi_4 \, e^{i\int d^4x \, \mathcal{L}_I} \rangle = \langle \phi_1 \cdots \phi_4 \rangle - i \int d^4x \frac{\lambda}{4!} \langle \phi_1 \cdots \phi_4 \, \phi_x^4 \rangle$$

$$- \frac{1}{2!} \int d^4x \int d^4y \left(\frac{\lambda}{4!}\right)^2 \langle \phi_1 \cdots\cdots \phi_4 \, \phi_x^4 \, \phi_y^4 \rangle$$

At $O(\lambda)$, we have



4! contractions

Two conventions for Feynman rules

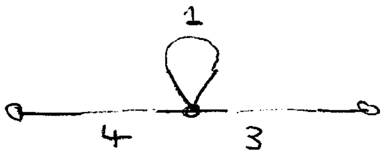
A.) The Feynman rule is $-i\frac{\lambda}{4!}$ and we have to count the number of contractions

B.) The Feynman rule is $-i\lambda$. Sometimes we have to divide by a symmetry factor because not all contractions arise.

As far as I know only (B.) is used in the QCD literature.

Example :

A.)



$$-i\frac{\lambda}{4!}4\cdot 3 = -i\frac{\lambda}{2}$$

B.)

$$-i\lambda\cdot\frac{1}{2}$$ ← symm. factor

Most of the time, diagrams are calculated in momentum space. To obtain the Feynman rules one Fourier transforms the Lagrangian

incoming momenta

$$-\frac{\lambda}{4!}\int d^4x\,\phi^4(x) = -\frac{\lambda}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\int d^4x\, e^{-i(k_1\dots k_4)x}\,\tilde{\phi}(k_1)\dots\tilde{\phi}(k_4)$$

$$\left[\int_k = \int\frac{d^4k}{(2\pi)^4}\right]$$

$$= -\frac{\lambda}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}(2\pi)^4\,\delta(k_1+k_2+k_3+k_4)\,\tilde{\phi}(k_1)\dots\tilde{\phi}(k_1)$$

At tree-level all momentum integrations can be performed trivially, eliminating δ-functions.

At higher orders nontrivial integrations over loop momenta remain.

To obtain the QCD Feynman rules, we have to Fourier transform the Lagrangian.

Let's first split it into $\mathcal{L}_0 + \mathcal{L}_{Int}$

$$\mathcal{L} = -\frac{1}{4}(F^a_{\mu\nu})^2 + \sum_q \bar{\psi}_q (i\not{\partial} - m_q)\psi_q - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$$

$$+ \bar{c}(-\partial^\mu D^{ab}_\mu)c$$

$$iD_\mu = i\partial_\mu + g A^a_\mu t^a$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g A^b_\mu A^c_\nu f^{abc}$$

$$D^{ab}_\mu = \partial_\mu \delta^{ab} + g f^{abc} A^c_\mu$$

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2$$

$$+ \sum_q \bar{\psi}_q (i\not{\partial} - m_q)\psi_q + \bar{c}(-\partial^2)c$$

The free propagators for these fields are obtained by Fourier transforming and inverting.

Remember that $i\partial_\mu \hat{=} k_\mu$.

$$\langle \psi^a_{i\alpha}(x)\, \bar\psi^a_{j\beta}(y) \rangle = \int_k \left(\frac{i}{\not k - m_q + i\varepsilon} \right)_{\alpha\beta} \delta_{ij}\, e^{-ik(x-y)}$$

$$\langle A^a_\mu(x) A^b_\nu(y) \rangle = \int_k \frac{i}{k^2 + i\varepsilon} \left(-g_{\mu\nu} + (1-\xi) \frac{k_\mu k_\nu}{k^2} \right) \delta^{ab}\, e^{-ik(x-y)}$$

... same as photon prop.

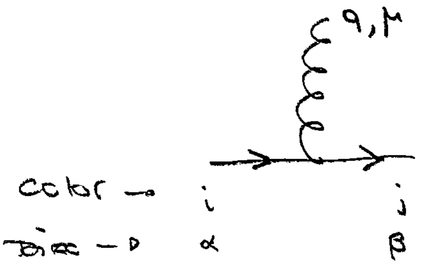
$$\langle c^a(x)\, \bar c^b(y) \rangle = \int_k \frac{i}{k^2}\, \delta^{ab}\, e^{-ik(x-y)}$$

All propagators are color diagonal.

Now let's look at the interactions

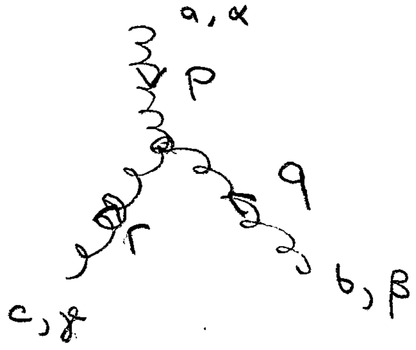
$$\mathcal{L}_{Int} = g A^a_\mu \bar\psi \gamma^\mu t^a \psi - g f^{abc} (\partial^\beta A^a_\alpha) A^b_\beta A^c_\alpha$$
$$- \frac{g^2}{4} f^{abe} A^a_\alpha A^b_\beta f^{cde} A^c_\beta A^d_\gamma$$
$$- \bar c^a g f^{abc} \partial^\mu A^c_\mu c^b$$

The first term gives the quark-gluon vertex



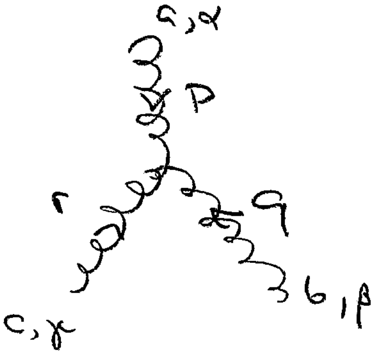
$$ig\,(\gamma^\mu)_{\beta\alpha}\,(t^a)_{ji}$$

The second is an interaction among three gluons, its Fourier transform is



$$-ig\,\frac{p^\beta}{i}\,g_{\alpha\gamma}$$

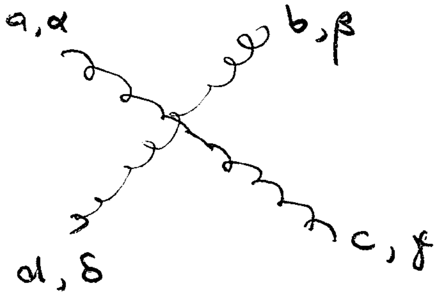
we follow convention B.), so the Feynman rule is the sum over 3! contractions, which correspond to the 3! permutations of the three gluon fields



$$g\,f^{abc}\left[(p-q)^\gamma g^{\alpha\beta}+(q-r)^\alpha g^{\beta\gamma} + \underset{\uparrow}{(r-p)}{}^\beta g^{\gamma\alpha}\right]$$

this is the above term.

Now, the four gluon term



Fourier transform is

$$-\frac{g^2}{4} f^{abe} f^{cde} g_{\alpha\gamma} g_{\beta\delta}$$

In this case, there are 4! permutations, of which always 4 are equivalent.

- Feynman rule

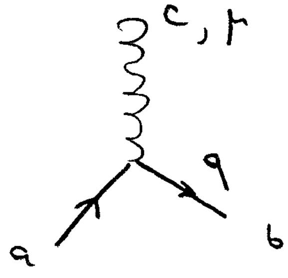
$$-ig^2 f^{abe} f^{cde} (g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma})$$

$$-ig^2 f^{ace} f^{bde} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\beta\gamma})$$

$$-ig^2 f^{ade} f^{bce} (g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\gamma} g_{\beta\delta})$$

- Finally, for the gluon-ghost coupling, we find

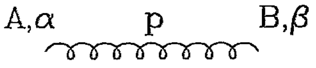
$$-ig f^{abc} \frac{q^\mu}{i}$$



The complete set of Feynman rules are given on the next page.

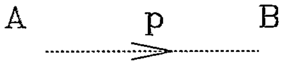
3.18.

[From K. Ellis, but with $g \to -g$]

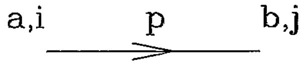


our ξ

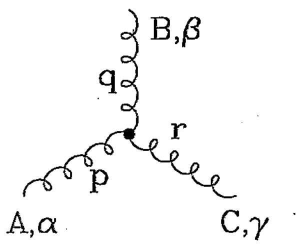
$$\delta^{AB}\ \left[-g^{\alpha\beta}+\overset{\downarrow}{(1-\lambda)}\ \frac{p^{\alpha}p^{\beta}}{p^{2}+i\epsilon}\right]\ \frac{i}{p^{2}+i\epsilon}$$



$$\delta^{AB}\ \frac{i}{(p^{2}+i\epsilon)}$$

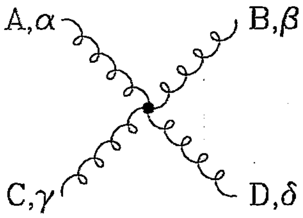


$$\delta^{ab}\ \frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

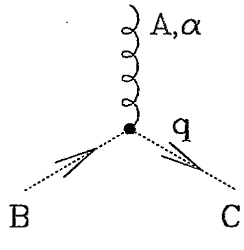


$$+g\ f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

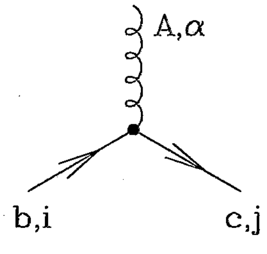
(all momenta incoming, $p+q+r = 0$)



$$\begin{aligned}&-ig^{2}\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]\\&-ig^{2}\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]\\&-ig^{2}\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]\end{aligned}$$



$$-g\ f^{ABC}q^{\alpha}$$



$$+ig\ (t^{A})_{cb}\ (\gamma^{\alpha})_{ji}$$

3.3. Dimensional Regularization

The Feynman diagrams which appear in the perturbative series contain UV divergent loop integrals. To make sense of these integrals, one needs to regularize the theory. If the divergencies can be absorbed into a redefinition of the parameters of the theory (couplings, masses, gauge parameters, ...), physical predictions will be meaningful, despite the presence of these divergences. This process is called renormalization.

It is desirable that the regularization preserves the symmetries of the theory. If not, one has to carefully recover the symmetry in the renormalization process, which can be very difficult.

It is especially important, that gauge symmetry is respected by the regularization. To my knowledge only to regularizations achieve this

1.) lattice regularization à la Wilson

2.) dimensional regularization
('t Hooft and Veltman '72)

Lattice regularization breaks Lorentz invariance, but leads to a nonperturbative definition of the theory.

Dim. reg. respects all symmetries, except chiral symmetry (problems with γ_5 & $\varepsilon_{\mu\nu\rho\sigma}$).

The idea of dim. reg. is simple to state: evaluate all loop integrals in d dimensions. For small enough d, the integrals will converge.

Let us do a sample calculation to illustrate the technique.

$$k^0 \to i k_E^0$$

$$\int \frac{d^dk}{(2\pi)^d} \frac{(k^2)}{(m^2 - k^2 - i\varepsilon)^n} = i \int \frac{d^dk_E}{(2\pi)^d} \frac{1}{(m^2 + k^2)^n}$$

$$= i \int \frac{d\Omega_d}{(2\pi)^d} \int_0^\infty dk_E \, k_E^{d-1} \frac{1}{(k_E^2 + m^2)^n}$$

The area of a d-dim sphere is

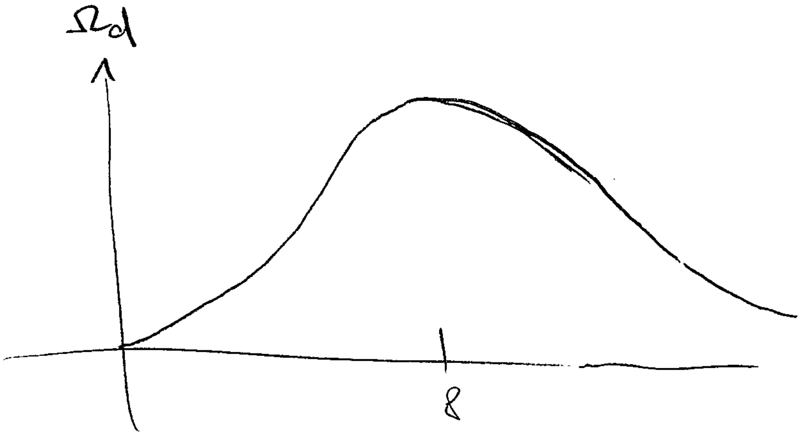
$$(\sqrt{\pi})^d = \left(\int dx \, e^{-x^2} \right)^d = \int d^dx \exp\left(-\sum_{i=1}^{d} x_i^2\right)$$

$$= \int d\Omega_d \int_0^\infty dx \, x^{d-1} e^{-x^2} = \int d\Omega_d \int_0^\infty dx^2 \, \frac{1}{2} (x^2)^{\frac{d}{2}-1} e^{-x^2}$$

$$= \left(\int d\Omega_d \right) \frac{1}{2} \Gamma(d/2)$$

$$\Rightarrow \quad \Omega_{d/2} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

d	Ω_d
1	2
2	2π
3	4π
4	$2\pi^2$
∞	0



Also the k-integral gives a Γ-function

$$\int_0^\infty dk\, k^{d-1} \frac{(k^2)^\alpha}{(k^2+m^2)^\beta} = (m^2)^{\alpha-\beta+\frac{d}{2}} \frac{\Gamma(\frac{d}{2}+\alpha)\Gamma(\beta-\frac{d}{2}-\alpha)}{2\Gamma(\beta)} \quad (*)$$

$$\int_0^\infty d\hat{k}\, k^\gamma (k^2+1)^\beta = \frac{1}{2}\int_0^1 dx\, (1-x)^{\overset{B-1}{\frac{\gamma-1}{2}}}\, x^{\overset{A-1}{\beta-2+\frac{1-\gamma}{2}}}$$

$$x = \frac{1}{k^2+1} \quad ; \quad k^2 = \frac{1-x}{x} \Big] = \frac{1}{2} B\left(\frac{\gamma+1}{2}, \beta-\frac{\gamma+1}{2}\right)$$

$$\int_0^1 dx\, x^\alpha (1-x)^\beta = B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Note that the integral on the LHS of (*)

is only defined if

$$d + 2\alpha < 2\beta$$

$$d + 2\alpha > 0$$

However, the expression on the RHS is well defined for arbitrary complex d, except for poles at $\frac{d}{2}+\alpha = 0, -1, -2, \dots$

and

$$\beta-\alpha-\frac{d}{2} = 0, -1, -2, \dots$$

Another interesting property is that the RHS vanishes for $\beta \to 0$. We now <u>define</u> the integral by the RHS:

$$(**) \quad \int d^d k \, \frac{(k^2)^\alpha}{(m^2-k^2-i\varepsilon)^\beta} := i \pi^{d/2} (m^2)^{d/2+\alpha-\beta} \frac{\Gamma(\alpha+\frac{d}{2})\Gamma(\beta-\alpha-\frac{d}{2})}{\Gamma(d/2)\Gamma(\beta)}$$

This implies, e.g.

$$\int d^d k \, (k^2)^\alpha = 0$$

For one-loop integrals involving several propagators, we can use the Feynman parameterization to combine them to the form

$$\int d^d k \, \frac{\{1, k_\mu, k_\mu k_\nu, \dots\}}{(M^2-k^2-i\varepsilon)^n} \qquad g^\mu{}_\mu = d$$

$$= \int d^d k \, \frac{\{1, 0, k^2 \frac{g^{\mu\nu}}{d}, \dots\}}{(M^2-k^2-i\varepsilon)^n}$$

where M depends on the Feynman parameters, external momenta, and masses.

Feynman parameterizations:

$$\int_0^1 dx \frac{1}{[Ax + B(1-x)]^2} = \frac{1}{AB}$$

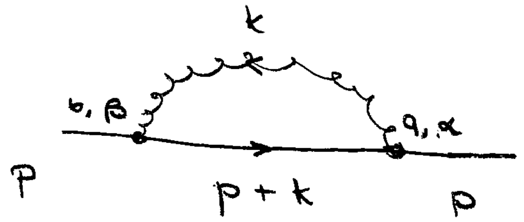
$$\int_0^1 dx_1 \dots dx_n \, \delta(\Sigma x_i - 1) \frac{\Pi x_i^{m-1}}{[\Sigma x_i A_i]^{\Sigma m_i}} \cdot \frac{\Gamma(m_1 + \dots + m_n)}{\Gamma(m_1) \dots \Gamma(m_n)}$$

$$= \frac{1}{A_1^{m_1} A_2^{m_2} \dots A_n^{m_n}}$$

In collider physics calculations, we encounter not only UV, but also infrared divergences, as we'll see soon. Also in these cases dim. reg. is used as a regulator.

A sample diagram

Consider the quark self-energy



The symmetry factor is 1 ; 2 contractions and $\frac{1}{2!}$ because we have two identical vertices. Let's calculate the amputated diagram, i.e. the diagram without external propagators.

$$-i\Sigma = \int \frac{d^dk}{(2\pi)^d} \left(ig\gamma^\alpha t^a \right) \cdot \frac{i}{\not{p} + \not{k} - m_q + i\varepsilon} \left(ig\gamma^\beta t^b \right)$$

$$\cdot \frac{i}{k^2 + i\varepsilon} \left[-g_{\alpha\beta} + \frac{k_\alpha k_\beta}{k^2 + i\varepsilon} (1 - \xi) \right] \delta^{ab}$$

Color structure: $t^a t^b \delta^{ab} = t^a t^a = C_F \mathbb{1}$

For simplicity let's use Feynman gauge $\xi = 1$.

$$-i\Sigma = -g^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2[(p+k)^2-m^2]}$$

$$\cdot \gamma^\alpha [\not{p} + \not{k} + m] \gamma_\alpha$$

Dirac structure:

$$\gamma_\alpha \gamma^\alpha = \frac{1}{2}\{\gamma_\alpha, \gamma^\alpha\} = g_\alpha{}^\alpha = d$$

$$\gamma_\alpha \gamma^\mu \gamma^\alpha = \gamma_\alpha \{\gamma^\mu, \gamma^\alpha\} - \gamma_\alpha \gamma^\alpha \gamma^\mu$$

$$= \gamma_\alpha 2 g^{\mu\alpha} - d\gamma^\mu = (2-d)\gamma^\mu$$

$$-i\Sigma = -g^2 C_F \int \frac{d^dk}{(2\pi)^d} \frac{(2-d)(\not{p}+\not{k}) + d \cdot m}{k^2[(p+k)^2-m^2]}$$

This expression contains two integrals

$$\{I, I_\mu\} = \int d^dk \frac{\{1, (p+k)^\mu\}}{k^2[(p+k)^2-m^2]}$$

$$\{I, I_\mu\} = \int_0^1 dx \int d^dk \; \frac{\{1, (p+k)^\mu\}}{[\underbrace{k^2 + 2xp.x}_{(k+xp)^2 - x^2p^2} + xp^2 - xm^2]}$$

shift $k \to k - xp$

$$\{I, I_\mu\} = \int_0^1 dx \int d^dk \; \frac{\{1, \; p^\mu - xp^\mu + \cancelto{0\ (\mathrm{odd})}{k^\mu}\}}{[k^2 - M^2]^2}$$

$$\text{where} \quad M^2 = +xm^2 - x(1-x)p^2$$

In this form, we can use our previous result for the k-integration.

$$\{I, I_\mu\} = \int_0^1 dx \, \{1, (1-x)p^\mu\} \; i\pi^{d/2} \, \Gamma(2 - d/2)$$
$$\cdot \left(xm^2 - x(1-x)p^2\right)^{d/2-2}$$

$d=4-2\varepsilon$

$$= i\pi^{d/2} \left(\frac{1}{\varepsilon} - \gamma_E\right) \int_0^1 dx \, \{1, (1-x)p^\mu\}$$
$$\times \left(xm^2 - x(1-x)p^2\right)^{-\varepsilon}$$

$$= i\pi^{d/2} \, \frac{1}{\varepsilon} \left\{1, \frac{p^\mu}{2}\right\} + \ldots.$$

We can plug this back in to get the divergent part of the quark self-energy in Feynman gauge $\xi = 1$.

$$-i\Sigma(p,m) = -i\left(\frac{\alpha_s}{4\pi}\right)C_F\left[-\frac{1}{\varepsilon}\not{p} + \frac{4}{\varepsilon}m\right] + \cdots$$

$\alpha_s = \frac{g^2}{4\pi}$, in analogy to $\alpha = \frac{e^2}{4\pi}$

For $\alpha_s \to \alpha$, $C_F \to 1$, we recover the QED result.

3.4 From Green's functions to cross sections

We have derived the Feynman rules for vacuum expetation values of time-ordered products of fields. From these, one can extract the scattering amplitudes.

The propagation of a particle manifests itself as a pole in the Fourier transformed correlation function, e.g.

$$\int d^4x \left\langle \phi(x)\, \phi(0) \right\rangle = \frac{i \overset{\text{on-shell wave function renorm.}}{Z}}{p^2 - m_{phys} + i\varepsilon} + \dots$$

To show this, use

$$\langle \phi(x)\, \phi(0) \rangle = \langle 0 | \phi(x)\, \phi(0) | 0 \rangle\, \Theta(x^0) + \langle 0 | \phi(0)\, \phi(x) | 0 \rangle\, \Theta(-x^0)$$

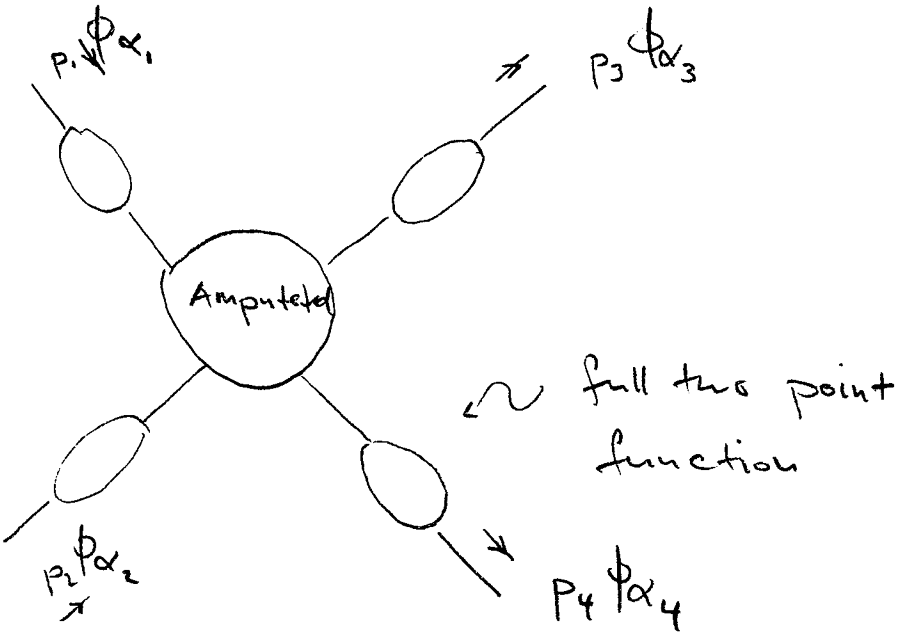
and insert a complete set of states

$$\sum_X |X\rangle\langle X| = |0\rangle\langle 0| + \int \frac{d^3p}{(2\pi)^3\, 2E} |p\rangle\langle p| + \dots$$

one finds $\sqrt{Z} = \langle 0|\phi(0)|p\rangle$.

For a free theory $m_{phys} = m$, $Z = 1$.

n-point correlation functions have the form



The indices $\alpha_1, \dots, \alpha_4$ are the Dirac or Lorentz indeces of the fields $\phi_{\alpha_i} = A_{\alpha_i}, \psi_{\alpha_i}$ or $\bar{\psi}_{\alpha_i}$

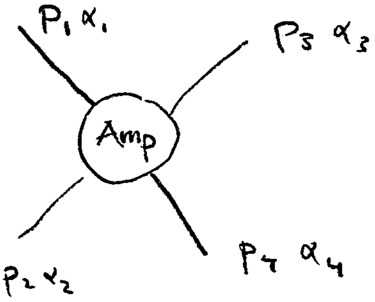
The scattering amplitude, or S-matrix, is given by

$$\langle p_3, s_3 ; p_4, s_4 \,|\, S \,|\, p_1, s_1 ; p_2, s_2 \rangle =$$

$$\sum_{\{\alpha\}} \langle p_3, s_3 | \phi_{\alpha_3}(0) | 0 \rangle \langle p_4, s_4 | \phi_{\alpha_4}(0) | 0 \rangle$$

$$\langle 0 | \phi_{\alpha_1}(0) | p_1, s_1 \rangle \langle 0 | \phi_{\alpha_2}(0) | p_2, s_2 \rangle$$

$*$



We have

These can be determined by calculating the full propagator

$$\langle 0 | \psi_\alpha(0) | p, s \rangle - \sqrt{Z_\psi}\, u_\alpha(p, s)$$

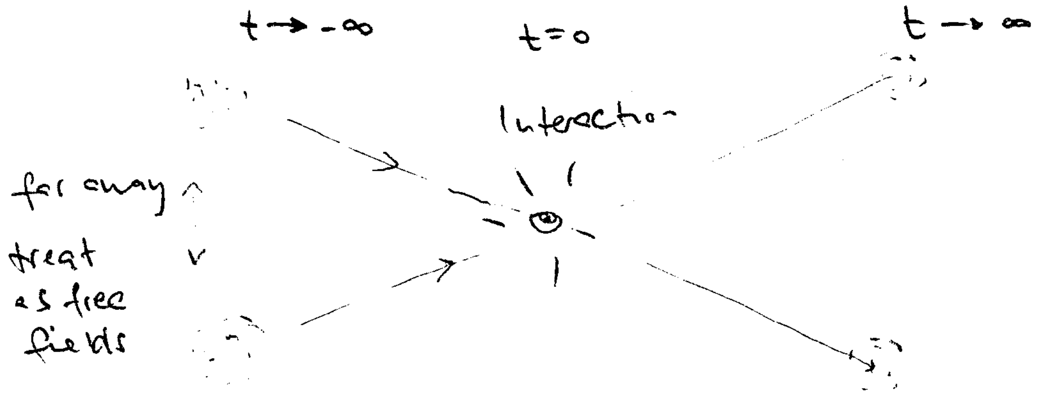
$$\langle 0 | \bar{\psi}_\alpha(x) | p, s \rangle = \sqrt{Z_\psi}\, \bar{v}_\alpha(p, s)$$

$$\langle 0 | A^a_\mu | p, \lambda, b \rangle = \sqrt{Z_A}\, \varepsilon_\mu(p, \lambda)\, \delta^{ab}$$

[I have not written out the color quantum numbers of the quarks, it would look like in the gluon case

This result is called the LSZ reduction formula. It's derivation is quite subtle.

To derive it one takes the Fourier transform of the correlation function, but to keep the different particles separated one has to work with wave packets



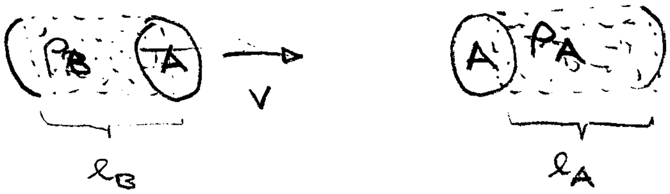
For theories with massless fields, the assumptions going into the derivation cannot really be fulfilled, e.g. an electron will always emit soft photons, so we cannot really prepare a wave packet with just an electron inside. In QCD things are even worse: how do you prepare a wave packet with a single quark inside?

There is a punishment for using the LSZ formula naively for QED and QCD: the scattering amplitudes contain infrared singularities.

In suitably defined observables, these IR singularities will be absent. Finding such "suitably" defined observables is what will keep us busy for much of the rest of the lecture.

Now that we have the scattering amplitude, we can calculate scattering cross sections.

Take two bunches of particles at a collider



Define $$\sigma = \frac{\text{" number of events "}}{\rho_A \, l_A \, \rho_B \, l_B \, A}$$

where A is the overlap area.

$$[\sigma] = \frac{1}{\frac{1}{m^3} \, m \, \frac{1}{m^3} \, m \, m^2} = m^2$$

The cross section has units of an area.

Assuming that both beams have area A and overlap completely, we can also write

$$N = \text{" Number of events "} = \frac{N_A \, N_B}{A} \cdot \sigma$$

Instead of cm^2, cross sections are usually given in barns : $1b = 1$ "barn" $= 10^{-24}\ m^2$

However cross sections measured at present colliders are typically in the $fb = 10^{-15}$ barn range.

An important quantity is the luminosity $\mathcal{L}$

$$\frac{dN}{dt} = \mathcal{L}\,\sigma .$$

In our example $\frac{N_A \cdot N_B}{A} = \int dt\ \mathcal{L}$

is called the integrated luminosity. It is measured in inverse femtobarn. $(fb)^{-1}$.

So with $7 (fb)^{-1}$ integrated luminosity, you'll get 7000 events for a 1pb cross section.

Run II at the Tevatron has collected around $7\ fb^{-1}$ as of now.

See slides for some measured cross sections.

To calculate the cross section one splits the S-matrix into $S = \mathbb{1} + iT$.

The $\mathbb{1}$ operator is relevant for the part of the S-matrix in which no scattering happens.

Then one defines

$$\langle p_1, \dots, p_n | iT | k_1, k_2 \rangle = (2\pi)^4 \delta(k_1 + k_2 - \sum_{i=1}^{n} p_i) \\ i\mathcal{M}(k_1, k_2 \to p_1, \dots, p_n)$$

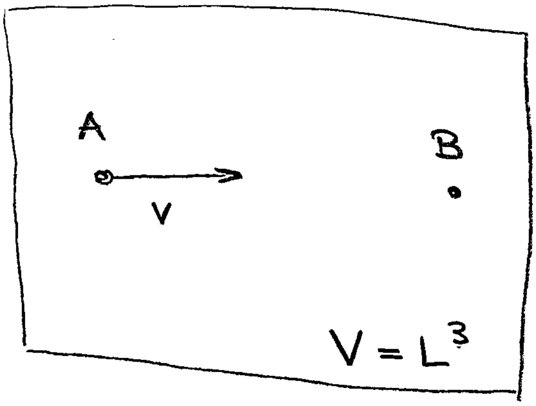
The differential cross section is given by

$$d\sigma = \frac{1}{4\sqrt{(k_1 \cdot k_2)^2 - m_1^2 m_2^2}} \left(\prod_{i=1}^{n} \frac{d^3 p_i}{2E_i (2\pi)^3} \right) (2\pi)^4 \delta(k_1 + k_2 - \Sigma p_i) \\ |\mathcal{M}(k_1, k_2 \to p_1, \dots p_n)|^2$$

For a careful derivation of this formula, one needs to consider wave packets.

Let's avoid this by considering scattering in a box of volume V during a time period T.

Let's assume that the box contains only two particles:



Either there will be scattering or not. The probability for scattering to some final state is

$$P = \frac{|\langle p_1, \dots, p_n | iT | k_A, k_B \rangle|^2}{\langle k_A | k_B \rangle \langle k_2 | k_2 \rangle \prod_{i=1}^{n} \langle p_i | p_i \rangle}$$

Here $P \equiv$ "number of events"

We had defined

$$\sigma = \frac{\text{" number of events "}}{\rho_A \, \ell_A \, \rho_B \cdot \ell_B \, A}$$

$$= \frac{P}{\frac{1}{V} L \frac{1}{V} \cdot V} = \frac{P}{\frac{L}{V}} = \frac{P}{\frac{|\vec{v}_A| T}{V}} \Big\} \text{flux}$$

Let's now evaluate P

$$P = \frac{(2\pi)^4 \delta(k_A + k_B - \sum_i p_i) \, (2\pi)^4 \delta^4(0) \, |\mathcal{M}|^2}{2E_A (2\pi)^3 \delta^3(0) \, 2E_B (2\pi)^3 \delta^3(0) \prod_{i=1}^{n} (2\pi)^3 \, 2E_i \, \delta^3(0)}$$

$$(2\pi)^3 \delta^3(0) = \int d^3x \, e^{i 0 \cdot x} = V \qquad (2\pi)^4 \delta^4(0) = V \cdot T$$

$$= \frac{(2\pi)^4 \delta(k_A + k_B - \sum_i p_i) \, |\mathcal{M}|^2 \cdot V \cdot T}{4 E_A E_B \, V^2 \prod_{i=1}^{n} 2E_i \, V}$$

This is the probability for scattering into a state with definite momentum.

Let's sum over momenta in some range.

In our box, the momenta are quantized

$\vec{k}_i = \frac{2\pi}{L} \vec{n}_i$, where $\vec{n}_i$ is a vector of

integers. For a large volume $V = L^3$

$$\sum_{n_i} = \frac{V}{(2\pi)^3} \int d^3 k_i$$

$$dP = \frac{(2\pi)^4 \delta(k_A + k_B - \sum_i p_i) |\mathcal{M}|^2}{4 E_A E_B \cdot V / T} \cdot \prod_{i=1}^{n} \frac{d^3 k_i}{2E_i (2\pi)^3}$$

$$d\sigma = \frac{dP \cdot V}{|\vec{v}_A| \cdot T} \qquad \left[E_B = m_B \qquad v = \frac{|\vec{p}_A|}{E_A} \right]$$

$$= \frac{(2\pi)^4 \delta(k_A + k_B - \sum_i p_i) |\mathcal{M}|^2}{4 m_B \cdot |\vec{p}_A^{\,\text{lab}}|} \prod_{i=1}^{n} \frac{d^3 k_i}{2E_i (2\pi)^3}$$

Note $|\vec{p}_A| m_B = \sqrt{(k_A \cdot k_B)^2 - m_A^2 m_B^2}$

With the expression on the RHS, we recover the cross section in an arbitrary frame.

3.5 The β-function and the running of α_s

The correlation functions of QCD contain UV divergences, which we are regulating by keeping $d \neq 4$. As we have seen in the example of the fermion self-energy, the divergences manifest themselves as $\frac{1}{\varepsilon}$ poles, where $d = 4 - 2\varepsilon$.

In order to eliminate the divergences into the parameters of the theory, one defines

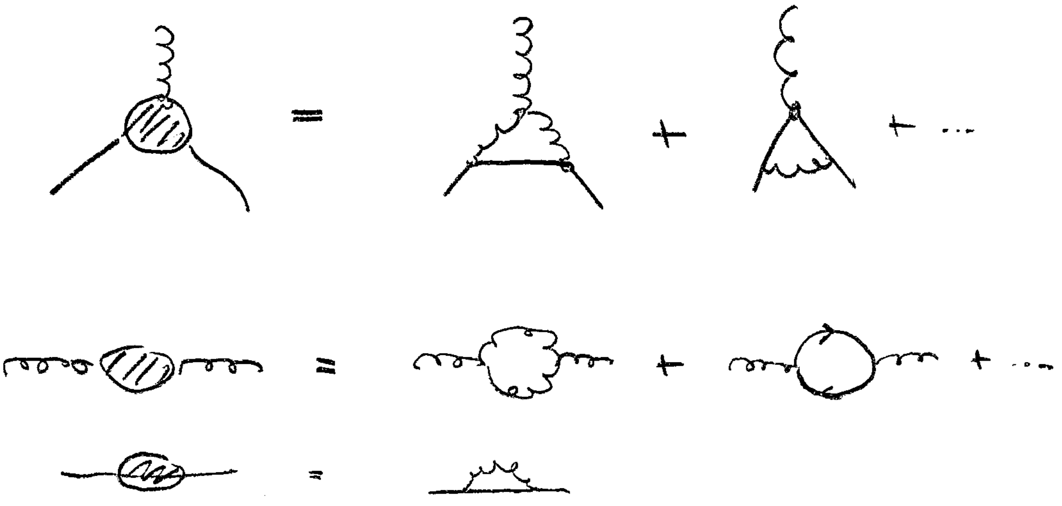
$$A^a_{0\mu} = Z_3^{1/2} A^a_\mu \qquad q_0 = Z_q^{1/2} q$$

$$\underset{\nearrow \text{ bare}}{g_0} = Z_g \underset{\nwarrow \text{ renormalized.}}{g_s \mu^\varepsilon} \qquad m_0 = Z_m m$$

(+ gauge parameter, ghost field)

The renormalization scale μ is arbitrary. The factor μ^ε guarantees that g_s is dimensionless.

By calculating



one determines

"number of quarks"

$$Z_g = 1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{6}C_A - \frac{2}{6}n_f\right)\frac{1}{\varepsilon}$$

$$\alpha_s = g^2/4\pi$$

Note that we have decided <u>not</u> to include any finite higher order terms into the Z-factor. This is called "minimal subtraction" (MS).

We would like to study the behavior of $g(\mu)$ when we change the scale μ.

Define $$\mu \frac{d}{d\mu} g(\mu) = \frac{d}{d\ln\mu} g(\mu) = \beta(g(\mu), \varepsilon)$$

Using that the bare coupling is μ-independent, we can obtain the β-function from the Z_g-factor

$$\frac{d}{d\ln\mu}\, g_0 = 0 = \frac{d}{d\ln\mu}\, Z_g\, g\, \mu^{\varepsilon}$$

So

$$\left(\frac{d}{d\ln\mu} Z\right) g\mu^{\varepsilon} + Z\,\beta(g,\varepsilon)\,\mu^{\varepsilon} + \varepsilon\, Z\, g\, \mu^{\varepsilon} = 0$$

$$\Rightarrow \quad \beta(g,\varepsilon) = \underbrace{-g\,\frac{1}{Z}\,\frac{d}{d\ln\mu} Z}_{} - \varepsilon g$$

$$= \underbrace{\beta(g)}_{\text{4-d } \beta\text{-function}} - \varepsilon g$$

Note that $Z = 1 + \frac{1}{\varepsilon} Z_{g,1} + \frac{1}{\varepsilon^2} Z_{g,2} + \dots$

only depends implicitly on μ via $g(\mu)$:

$$\frac{d}{d\ln\mu} Z = \frac{dZ}{dg}\,\frac{dg}{d\ln\mu}$$

In the MS scheme in dimensional regularization the β-function can be obtained from the $\frac{1}{\varepsilon}$ part of the Z-factor:

$$\boxed{\beta(g) = 2g^3 \frac{dZ_{g,1}}{dg^2}} \qquad Z_g = 1 + \frac{1}{\varepsilon} Z_{g,1} + \frac{1}{\varepsilon^2} Z_{g,2} + \dots$$

To show this use

$$Z_g\, \beta(g) = -g \frac{dZ}{d\ln\mu} = -g \frac{dZ}{dg} \frac{dg}{d\ln\mu}$$

$$= -g \frac{dZ}{dg} \beta(g,\varepsilon)$$

$$= -g \frac{dZ}{dg} \left(\beta(g) - \varepsilon g \right)$$

Now take the $O(\varepsilon^0)$ coefficient of this equation

$$\beta(g) = +g^2 \frac{dZ_{g,1}}{dg} = 2g^3 \frac{dZ_{g,1}}{dg^2}$$

Using $Z_g = 1 - \frac{g^2}{16\pi^2}\left(\frac{11}{6}C_A - \frac{2}{6}n_f\right)\cdot\frac{1}{\varepsilon}$

we have

$$\beta(g) = -g\,\frac{g^2}{16\pi^2}\left(\frac{11}{3}C_A - \frac{2}{3}n_f\right) + O(g^5)$$

$$= -g\,\frac{\alpha_s}{4\pi}\,\beta_0$$

In terms of α_s

Gross & Wilczek '73 Politzer '73
Nobel price '99

$$\frac{d\alpha_s}{d\ln\mu} = -2\alpha_s\left[\beta_0\left(\frac{\alpha_s}{4\pi}\right) + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \beta_2\left(\frac{\alpha_s}{4\pi}\right)^3 + \beta_3\left(\frac{\alpha}{4\pi}\right)^3\right]$$

Ritbergen, Larin, Vermaseren '97

Note $\beta_0 = 11 - \frac{2}{3}n_f > 0$ for $n_f < 17$
$= 7$ in QCD with 6 quarks

In QED $\beta_0 = -\frac{2}{3}n_f < 0$

It turns out that only non-abelian gauge theories can have $\beta_0 < 0$. In fact, Gross & Wilczek wanted to prove that all QFTs have $\beta_0 < 0$. Fortunately they failed!

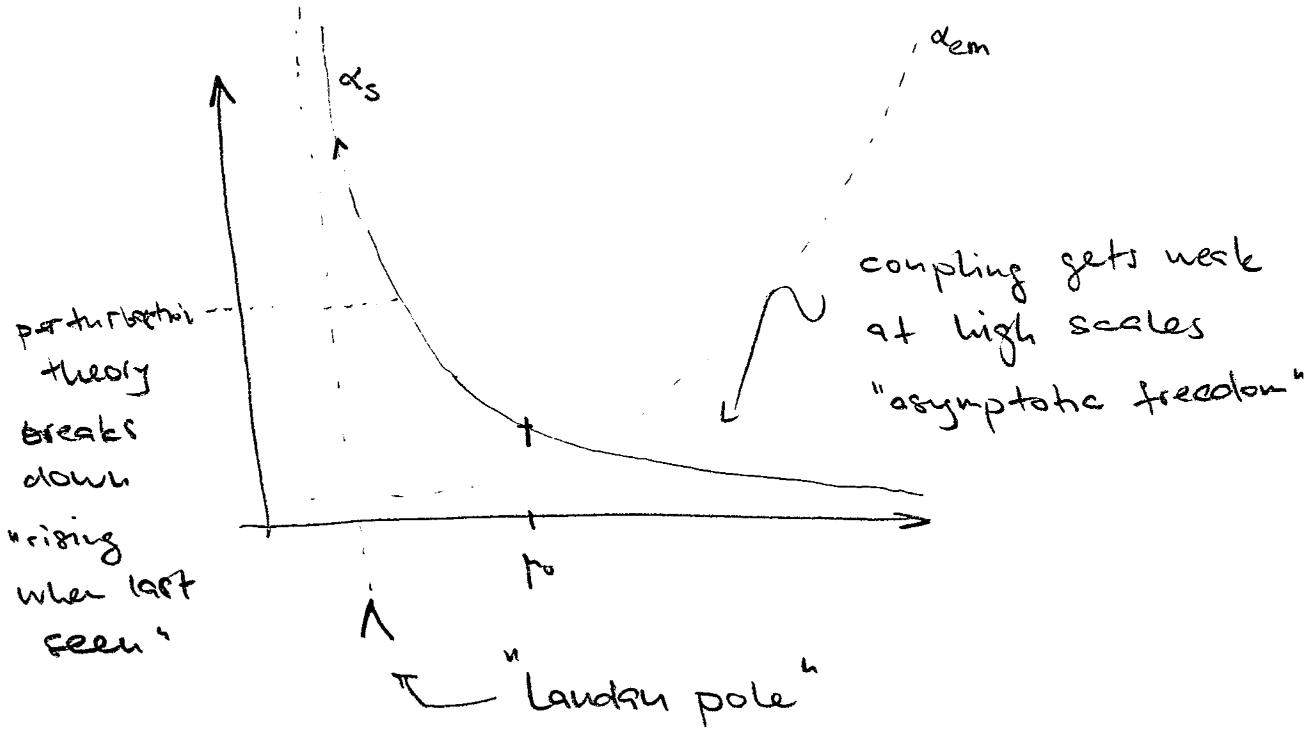
To understand the meaning of $\beta_0 > 0$, let's solve the RG equation for α_s

$$\frac{d}{d\ln\mu}\alpha_s = -2\alpha_s\beta_0\frac{\alpha_s}{4\pi} = \beta(\alpha_s)$$

$$\Rightarrow \int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)} = \int_{\mu_0}^{\mu} d\ln\mu = \ln\left(\frac{\mu}{\mu_0}\right)$$

$$\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha}{\beta(\alpha)} = -\frac{4\pi}{2\beta_0}\int_{\alpha(\mu_0)}^{\alpha(\mu)} \frac{d\alpha}{\alpha^2} = +\frac{4\pi}{2\beta_0}\left(\frac{1}{\alpha(\mu)} - \frac{1}{\alpha(\mu_0)}\right)$$

$$\leadsto \alpha(\mu) = \frac{\alpha(\mu_0)}{1 + \frac{\alpha(\mu_0)}{4\pi}\beta_0\ln\left(\frac{\mu^2}{\mu_0^2}\right)} \equiv \frac{4\pi}{\beta_0\ln\left(\frac{\mu^2}{\Lambda^2}\right)}$$



Note that the coupling not only depends on the renormalization scale μ, but also on the renormalization scheme.

The standard scheme is the $\overline{MS}$ scheme, in which some universal finite $\ln(4\pi) + \gamma_E$ factors are absorbed into the Z-factors.

$$\mu_{\overline{MS}} = \mu_{MS} \cdot e^{\gamma_E/2} (4\pi)^{-1/2}$$

$$\alpha_{MS}(\mu) = \alpha_{\overline{MS}}(\mu) \left(1 + \beta_0 (\gamma_E - \ln 4\pi) \frac{\alpha_{\overline{MS}}(\mu)}{4\pi}\right) + \dots$$

For a review of the determination of α_s, see S. Bethke 0908.1135.

4. $e^+e^- \to$ hadrons

The simplest quantity that can be evaluated perturbatively is the total cross section

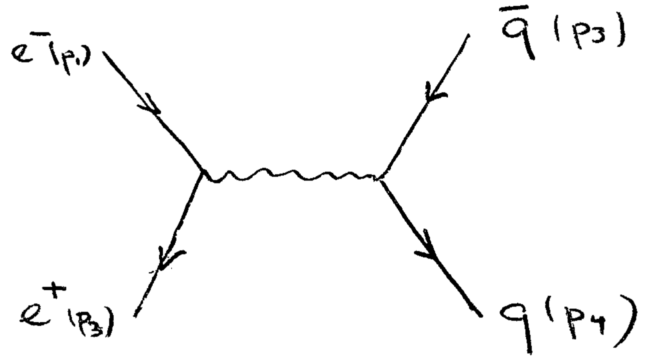
$e^+e^- \to$ hadrons.

In the following we will calculate this quantity, discuss the infrared singularities which appear at intermediate stages, and compare to data.

In chapter 5, we will then justify the use of perturbation theory.

4.1 $e^+e^- \to \bar{q}q$

To get the cross section, we evaluate the diagram



The sattering amplitude is $\quad s = (p_1+p_2)^2$

$$i\mathcal{M} = \Big(\overset{1+\frac{\alpha_s}{4\pi}C}{\sqrt{Z_e}}\Big)^2 \Big(\overset{1+\frac{\alpha_s}{4\pi}(\dots)}{\sqrt{Z_q}}\Big)^2 \; \bar{v}(p_2,m_e)(-ie\gamma^\mu)u(p_1,m_e)$$

$$\frac{i}{s}\left(-g_{\mu\nu} + (1-\xi)(p_1+p_2)^\mu (p_1+p_2)^\nu\right)$$

$$\bar{u}(p_4,m_q)(-iee_q\gamma^\nu)v(p_3,m_q)$$

$$i\mathcal{M} = +i\,\frac{e^2 e_q}{s}\,\bar{v}(p_2,m_e)\gamma^\mu u(p_1,m_e)$$

$$* \; \bar{u}(p_4,m_q)\gamma^\mu v(p_3,m_q)$$

Now calculate $|\mathcal{M}|^2$, sum over quark spins, average over electron spins, and sum over quark colors.

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{1}{4}\sum_{\text{spins}} \frac{e^4 e_q^2}{s^2}\; \bar{v}(p_2,m_e)\gamma^\mu u(p_1,m_e)\,\bar{u}(p_1,m_e)\gamma^\nu v(p_2,m_e)$$

$$\cdot\; \bar{u}(p_4,m_q)\gamma^\mu v(p_3,m_q)\;\bar{v}(p_3,m_q)\gamma^\nu u(p_4,m_q)$$

__Note:__ $\quad \sum_s u(p,s)\,\bar{u}(p,s) = \not{p} + m$

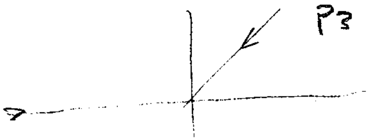
$\quad\quad\quad\;\; \sum_s v(p,s)\,\bar{v}(p,s) = \not{p} - m$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4 e_q^2}{4s^2}\, \mathrm{tr}\left[(\not{p}_2 - m_e)\gamma^\mu(\not{p}_1 + m_q)\gamma^\nu\right]$$

$$\circ\ \mathrm{tr}\left[(\not{p}_4 + m_q)\gamma^\mu(\not{p}_3 + m_q)\gamma^\nu\right]$$

$$= e^4 e_q^2\left[1 + \underset{\substack{\uparrow \\ \text{c.m.s sattnig angle}}}{\cos^2(\theta)}\right] + O(m_e^2, m_q^2)$$

$p_3 = \frac{\sqrt{s}}{2}(1, \sin\theta, 0, \cos\theta)$



$p_1 = \frac{\sqrt{s}}{2}(1, 0, 0, +1)$

$p_2 = \frac{\sqrt{s}}{2}(1, 0, 0, +1)$

$$(p_1 - p_3)^2 = -2\frac{s}{4}(1 - \cos\theta) = -\frac{s}{2}(1 - \cos\theta)$$

$$\delta(\sqrt{s} - E_3 - E_4)\,\delta^3(\vec{p}_3 + \vec{p}_4)$$

$$d\sigma = \int \frac{d^3p_3}{(2\pi)^3 2E_3}\int\frac{d^3p_4}{(2\pi)^3 2E_q}(2\pi)^4\,\delta^4(p_1 + p_2 - p_3 - p_4)$$

$$\frac{1}{4\sqrt{\underset{\frac{s}{2}}{(p_1\cdot p)^2} - m_e^4}}|\mathcal{M}|^2$$

$$= \int\frac{d^3p_3}{(2\pi)^2\, 4|\vec{p}_3|^2}\,\delta(\sqrt{s} - 2|\vec{p}_3|)\,\frac{1}{4\, s/2}|\mathcal{M}|^2$$

$\delta(a -)$

$$d\sigma = \int \frac{dp\, p^2}{(2\pi)^2 4p^2} \int_{-1}^{1} d\cos\theta \int_{0}^{2\pi} d\varphi \; \delta(\sqrt{s} - 2p_3)\, \frac{1}{2s}\, |M|^2$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s} |M|^2$$

Including the sum over spins, add the sum over colors, we have

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi s} (1 + \cos^2\theta) . e^4 N_c \sum_q e_q^2$$

$$e^4 = 16\pi^2 \alpha^2$$

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{2s} (1 + \cos^2\theta)\, N_c \sum_q e_q^2$$

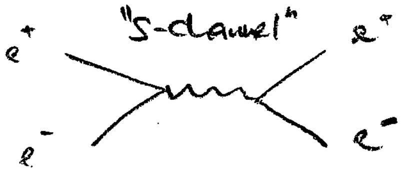
$$\boxed{\sigma^{tot} = \frac{\pi\alpha^2}{2s} \cdot \frac{8}{3}\, N_c \sum_q e_q^2}$$

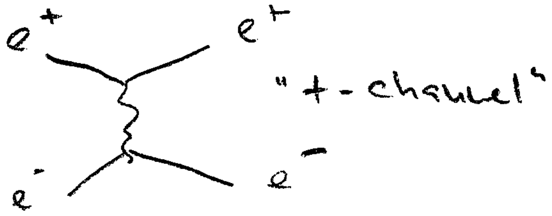
It is common to look at the ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

$$= N_c \sum_q e_q^2 \cdot \{1 + O(\alpha_s)\}$$

Note that $e^+e^- \to e^+e^-$ has two contributions

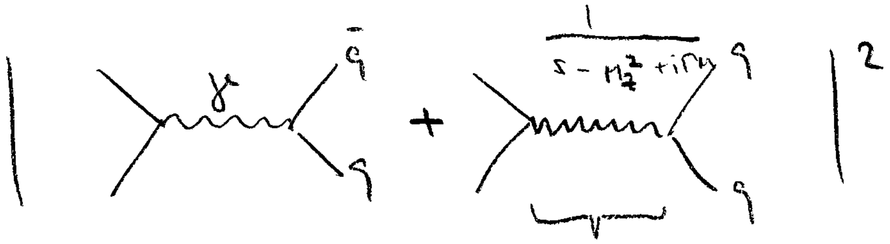




The t-channel diagram does not exist for $e^+e^- \to \mu^+\mu^-$

We have neglected contributions due to Z-exchange, which is fine at low energies

$|M|^2 =$



see Ellis's book for the full expressions.

$\sim s/M_Z^2$ for $q^2 \to 0$

We have also neglected quark (and electron) masses. Including the mass in our calculation is straightforward, but makes both the phase-space integrals and the amplitude more complicated.

In terms of the quark velocity $\beta = \frac{v_{c.m.}}{c} = \sqrt{1-\frac{4m_Q^2}{s}}$, one finds

$$R_{e^+e^-\to\bar{Q}Q} = N_c\, e_Q^2\, \beta\, \frac{3-\beta^2}{2}$$

$$= N_c\, e_Q^2 \left[1 - \frac{6m_Q^4}{s^2} + O\left(\frac{m_Q^6}{s^3}\right)\right]$$

so the masses of light quarks can be safely neglected. (If R would depend strongly on these masses, this would spell trouble, i.e. non-perturbative physics.)

Note that heavy quarks with $2m_Q > \sqrt{s}$ cannot be produced, so the sum over flavors in the R-ratio should not contain those.

Looking at the data*, the agreement is quite nice, except at very low energy or near thresholds $\sqrt{s} \cong 2m_Q$, where Q is a charm or bottom quark.

In fact, the R-ratio provides a great illustration of asymptotic freedom: the hadronic cross section at high energies is well approximated by the cross section e^+e^- to a pair of free quarks.

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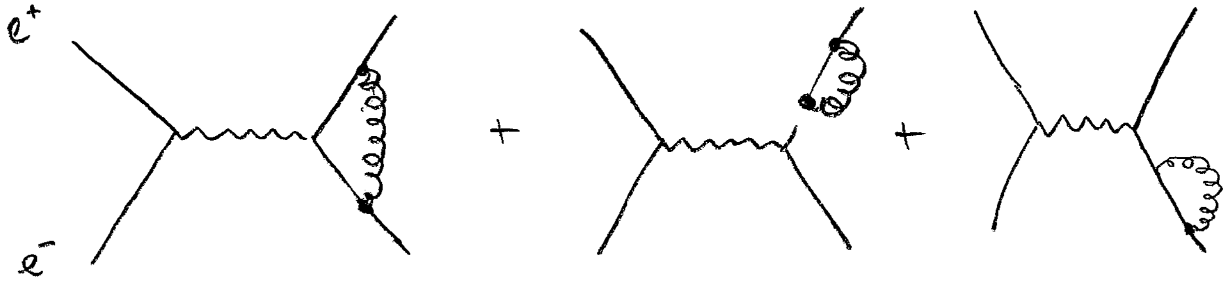
* See slides

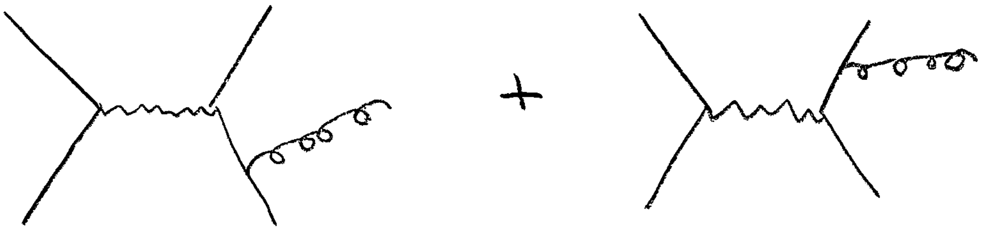
Let us now look at perturbative corrections.

At order α_s, there are two types of corrections:

1.) Virtual corrections: one-loop corrections to $e^+e^- \to q\bar{q}$



2.) Real emission diagrams



Their evaluation involves an interesting subtlety, which we'll discuss shortly, but let's first look at the result.

One finds that

$$R = N_c \sum_q e_q^2 \left\{ 1 + \frac{\alpha_s}{\pi} \right\}$$

The result is known to $O(\alpha_s^4)$! At the next order, evaluating the diagrams gives

$$R = N_c \sum_q e_q^2 \left\{ 1 + \frac{\alpha_s}{\pi} + \left(\frac{\alpha_s}{\pi}\right)^2 \cdot \left[C_2 + \frac{\beta_0}{4} \left(\underset{=\!=\,!}{\frac{1}{\varepsilon}} - \gamma_E + \ln(4\pi) - \ln(s) \right) \right] \right\}$$

with $C_2 = \left(\frac{2}{3}\zeta_3 - \frac{11}{12}\right) n_f + \frac{365}{24} - 11\zeta_3$

$= 1{,}99 - 0{,}115\, n_f$.

Here $\alpha_s \equiv \alpha_s^0$ the bare coupling.

Renormalize $\alpha_s^0 = \alpha_s(\mu) Z_g^2 \mu^{2\varepsilon}$

$= \alpha_s(\mu) \left[1 - \frac{\alpha}{4\pi} \frac{\beta_0}{2\varepsilon} \right]^2 \mu^{2\varepsilon}$

$$\mu = \mu_{\overline{MS}}\, e^{-\gamma_E/2} (4\pi)^{+1/2}$$

$$R = N_c \sum_q e_q^2 \left\{ 1 + \left(\frac{\alpha_s(\mu)}{\pi}\right) + \left(\frac{\alpha(\mu)}{\pi}\right)^2 \left[C_2 - \frac{\beta_0}{4} \ln \frac{s}{\mu^2} \right] \right\}$$

Note:

$$\frac{d}{d\ln\mu}\left(\frac{\alpha(\mu)}{\pi}\right) = \frac{1}{\pi}\left(-2\alpha_s(\mu)\beta_0 \frac{\alpha_s}{4\pi}\right) = \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left[-\frac{\beta_0}{2} \right]$$

$$\Rightarrow \frac{d}{d\ln\mu} R = N_c \sum_q e_q^2 \left\{ \left(\frac{\alpha}{\pi}\right)^2 \left(-\frac{\beta_0}{2}\right) + \frac{\beta_0}{2}\left(\frac{\alpha}{\pi}\right)^2 \right\} + O(\alpha^3)$$

$$= O(\alpha^3)$$

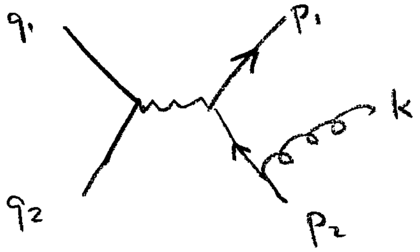
So R is scale invariant up to the order we have calculated.

However, if we choose $\mu^2 \gg s$ or $\mu^2 \ll s$, the $\ln\left(\frac{s}{\mu^2}\right)$ term will become very large and perturbation theory breaks down, so we should choose $\mu^2 \approx s$. Often people vary $\mu \in \left[\frac{\sqrt{s}}{2}, 2\sqrt{s}\right]$ to estimate the uncertainty from higher orders.

4.11

7.2. Soft and collinear divergences, IR safety

The calculation of the real emission diagrams



is straightforward (and tedious). One obtains

$$\frac{1}{4}\sum |\mathcal{M}|^2 = 24\, C_F\, e^4 e_q^2\, g_s^2\, \frac{(p_1\cdot q_1)^2 + (p_1\cdot q_2)^2 + (p_2\cdot q_1)^2 + (p_2\cdot q_2)^2}{q_1\cdot q_2\;\, p_1\cdot k\;\, p_2\cdot k}$$

Phase space:

Let's work in the CMS and parameterize phase space as

$$q_1 = \frac{\sqrt{s}}{2}(1, 0, 0, 1)$$

$$q_2 = \frac{\sqrt{s}}{2}(1, 0, 0, -1)$$

$$p_1 = x_1 \frac{\sqrt{s}}{2}(1, S_1, 0, C_1) \qquad ; \quad C_1 = \cos\theta_1, \ldots$$

$$p_2 = x_2 \frac{\sqrt{s}}{2}(1, S_2 C_\theta, S_1 S_\theta, C_2)$$

$$k = q_1 + q_2 - p_1 - p_2$$

Additional constraints: $x_1 = 0 \ldots 1$

$x_2 = 0 \ldots 1$

$x_3 = 2 - x_1 - x_2 = 0 \ldots 1 \quad \Rightarrow x_1 + x_2 \geq 1$

$$k^2 = 0 \quad \Rightarrow \quad C_\theta = \frac{2 - 2x_1 - 2x_2 + x_1 x_2 (1 - C_1 C_2)}{S_1 S_2 x_1 x_2}$$

Plug this into the phase-space integral, rewrite the integral as an integral over $\theta_1, \theta_2, x_1, x_2$

The angles θ_1 and θ_2 appear only in the numerator of the amplitude, since

$$k \cdot p_1 = \frac{s}{2}(1 - x_2) \qquad k \cdot p_2 = \frac{s}{2}(1 - x_1)$$

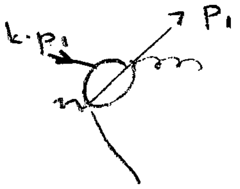
and are thus easily performed.

After lengthy algebra, one has

$$\sigma^{q\bar{q}g} = \sigma^{(0)} \int dx_1 \int dx_2 \; \frac{C_F \alpha_s}{2\pi} \; \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

The integration region is $0 \leq x_1 , x_2 \leq 1 \quad x_1 + x_2 \geq 1$

Unfortunately, this is ill-defined because of the $(1-x_1)^{-1}$ & $(1-x_2)^{-1}$ factors.



$$k \cdot p_1 = \frac{s}{2}(1-x_2) = E_g E_q (1 - \cos\theta_{qg})$$

$$k \cdot p_2 = \frac{s}{2}(1-x_1) = E_g E_{\bar{q}} (1 - \cos\theta_{\bar{q}g})$$

The singularities are from the region of phase-space where particles become collinear

$$\theta_{qg} \rightarrow 0 \quad \text{or} \quad \theta_{\bar{q}g} \rightarrow 0$$

or soft, $E_g \rightarrow 0$, $E_q \rightarrow 0$ or $E_{\bar{q}} \rightarrow 0$.

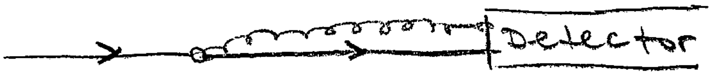
The collinear singularities are regulated (in this example) if the quark is massive

$$k \cdot p_1 = E_g (E_q - |\vec{p}_q| \cos\theta_{qg})$$

However, if the final result for the cross section would be $\propto \frac{1}{m_q}$ the perturbative results would not make sense, because $m_{u,d} \sim$ few MeV and PT no longer makes sense at such a low scale.

Furthermore, even with a quark mass, a soft singularity from $E_g \to 0$ remains.

The physical origin of these singularities is that it is not possible to distinguish a quark from a quark and a collinear gluon.



To get something sensible, we thus have to add $\sigma^{q\bar{q}} + \sigma^{q\bar{q}g}$ in the region where the gluon becomes collinear or soft.

For the total cross section we add all channels. $\sigma^{tot} = \sigma^{q\bar{q}} + \sigma^{q\bar{q}g}$, so this should be sensible. Indeed one finds that both σ^{qqg} and σ^{qq} have IR divergences at $O(\alpha_s)$ but they cancel in the total cross section

To get the cancellation one has to regularize the IR div's. To do so, one could introduce masses for quarks and gluons. A much more elegant method is to use dimensional regularization for both loop and phase-space integrals.

For the 3-body phase-space, we can use the same parameterization as before. However, angular integrations are now rewritten as

$$\int d^{d-1}p = \int dp\, p^{d-2} \underbrace{\int d\Omega_{d-1}}_{\text{solid angle}}$$

$$= \int dp\, p^{d-2} \int_{-1}^{1} d\cos\theta\, \sin^{d-4}\theta \int d\Omega_{d-2}$$

In d-dimensions, $d = 4-2\varepsilon$, one then finds

$$\sigma^{q\bar{q}g}(\varepsilon) = \sigma_0 \, H(\varepsilon) \int \frac{dx_1 dx_2}{P(x_1,x_2)} \frac{2\alpha_s}{3\pi} \left[\frac{(1-\varepsilon)(x_1^2+x_2^2)+2\varepsilon(1-x_3)}{(1-x_1)(1-x_2)} - 2\varepsilon \right]$$

$$P(x_1,x_2) = (1-x_1)^{\varepsilon}(1-x_2)^{\varepsilon}(1-x_3)^{\varepsilon}$$

$$x_3 = 2-x_1-x_2$$

$$H(\varepsilon) = \frac{3(1-\varepsilon)}{(3-2\varepsilon)\,\Gamma(2-2\varepsilon)} (4\pi)^{2\varepsilon} = 1 + O(\varepsilon)$$

For $\varepsilon < 0$, the integrals can be performed and one finds

$$\sigma^{q\bar{q}g}(\varepsilon) = \sigma_0 \frac{C_F \alpha_s}{2\pi} H(\varepsilon) \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} - \pi^2 + O(\varepsilon) \right]$$

↖ Two divergences:
soft & collinear

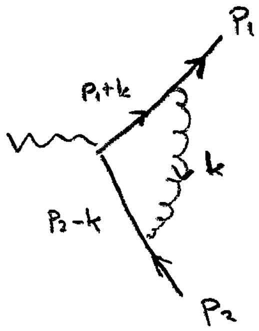
For the virtual corrections, one finds

$$\sigma^{q\bar{q}}(\varepsilon) = \sigma_0 \frac{C_F \alpha_s}{2\pi} H(\varepsilon) \left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \pi^2 \right]$$

Also in the virtual part, there are collinear and soft divergences.

They arise from regions of the loop integration where the loop momentum becomes soft or collinear with the external momentum.

E.g.



involves $I = \int d^dk \frac{1}{(p_1+k)^2 (p_2-k)^2 k^2}$

E.g. Soft divergence:

$$\hat{\int} d^dk \frac{1}{2p_1\cdot k \, 2p_2\cdot k \, k^2}$$ is logarithmically IR div.

In the sum of real and virtual corrections the IR singularities cancel:

$$\sigma^{q\bar{q}}(\varepsilon) + \sigma^{q\bar{q}g}(\varepsilon) = \sigma_0 \frac{C_F \alpha_s}{2\pi} \cdot \left[\frac{3}{2} \right] + O(\varepsilon)$$

$$= \sigma_0 \frac{\alpha_s}{\pi} .$$

An observable in which the IR singularities present in the amplitudes cancel is called infrared safe.

Since exclusive cross sections, such as $\sigma^{q\bar{q}}$ or $\sigma^{q\bar{q}g}$, are all IR divergent, such observables must be inclusive.

The total cross section is as inclusive as it gets and is IR safe. Can one find other IR safe observables?

This is possible and the main categories are

1.) Jet cross sections (as defined by a jet algorithm)

2.) Event shapes

These IR safe observables must be defined in such a way that physically indistinguishable final states are always included, i.e.

If the partonic state $|X\rangle$ contributes to the cross section, then also $|X$ + "soft gluons" + "collinear gluons & quarks" $\rangle$.

We'll come back to IR safety later when we discuss event shapes & jets.

5.) Operator product expansion in e^+e^-

The operator product expansion (OPE) is a very useful tool with many applications in QFT. In our case, it will allow us to show

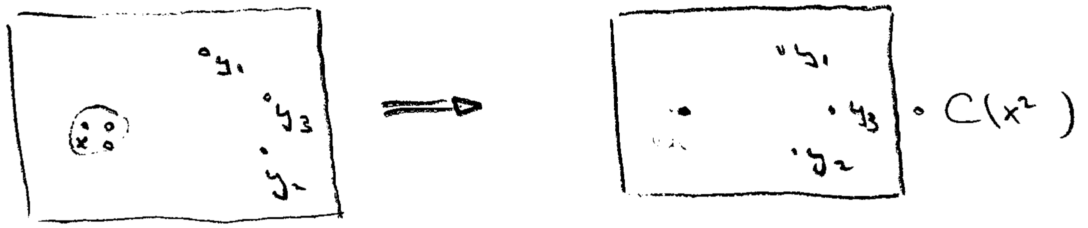
1.) that our calculation of the cross section in terms of quarks & gluons can be justified (in some cases), and

2.) what the nonperturbative corrections to our result are.

General idea: Consider product of two operators $\mathcal{O}_1(x)\mathcal{O}_2(0)$. Now look at Green's functions

$$G(x,0,y_1,\dots y_n) = \langle\!\langle \mathcal{O}_1(x)\mathcal{O}_2(0)\,\phi(y_1)\dots\phi(y_n)\rangle\!\rangle$$

In the limit $y_i^2 \gg x^2$ (in Euclidean space) we should be able to replace $\mathcal{O}_1(x)\mathcal{O}_2(0)$ by a local operator times a function of x^2.



Since this should hold for any Green's function, the relation should be an <u>operator relation</u>.

Wilson proposed '69

There are singularities in the limit $x \to 0$, absorb into C_n.

$$\mathcal{O}^{ren}(x)\,\mathcal{O}^{ren}(0) \underset{x\to 0}{=} \sum_n C_n(x^2;\mu)\,\mathcal{O}^{ren}_n(0)$$

The operators $\mathcal{O}_n$ have the same quantum numbers as the product and it is useful to order the operators by their dimension.

For example:

$$\phi(x)\phi(0) = C_1\cdot\mathbb{1} + C_{\phi^2}\,\phi^2(0) + C_{\phi^4}\,\phi^4 + C_{(\partial\phi)^2}\,\partial_\mu\phi(0)\partial^\mu\phi(0) + \dots$$

$\phi(x)$ has dimension of mass:

$$C_1 \sim \frac{1}{x^2} \qquad C_{\phi^2} \sim 1 \qquad C_{\phi^4} \sim x^2$$

The lowest dimensional operators have the most singular coefficients as $x \to 0$. The higher-dim. operators are suppressed.

We can thus approximate the operator product by the first few operators on the RHS.

The naive dimensional analysis is $\overset{\text{slightly}}{\vee}$ modified by renormalization, since $C_i = C_i(x^2, \mu)$ but as in the case of the coupling constant, the μ-dependence can be controlled using a RG equation. In perturbation theory higher order corrections will involve $\ln\left(\frac{x^2}{\mu^2}\right)$ terms.

5.1 The optical theorem

The optical theorem allows us to rewrite the total cross section as the imaginary part of the forward scattering amplitude. In this form, we will then apply the OPE.

- It follows from the unitarity of the S-matrix, $S^\dagger S = \mathbb{1}$.

Write $S = 1 + iT$,

$$(1 - iT^\dagger)(1 + iT) = 1$$

$$\rightarrow \quad \underbrace{-i(T - T^\dagger)}_{2\,\mathrm{Im}} = T^\dagger T$$

- Rewrite the RHS:

$$\langle p_1 p_2 | T^\dagger T | k_1, k_2 \rangle = \sum_X \langle p_1 p_2 | T^\dagger | X \rangle \langle X | T | k_1, k_2 \rangle$$

$$2\,\mathrm{Im}\, \mathcal{M}(k_1, k_2 \rightarrow p_1, p_2)$$

$$= \sum_X \mathcal{M}^*(p_1, p_2 \rightarrow p_X)\, \mathcal{M}(k_1, k_2 \rightarrow p_X)$$
$$(2\pi)^4 \delta(k_1 + k_2 - p_X)$$

To get the standard form set $k_1 = p_1$, $k_2 = p_2$

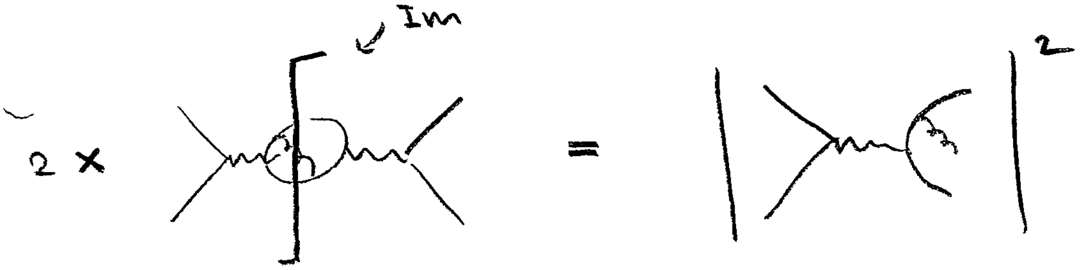
$$\Rightarrow \quad 2\,\mathrm{Im}\,\mathcal{M}(p_1, p_2 \to p_1, p_2)$$

$$= \sum_X |\mathcal{M}(p_1, p_2 \to p_X)|^2\, \delta^4(p_1 + p_2 - p_X)$$

$$= 4\underbrace{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}_{= E_{c.m.} \cdot p_{c.m.} \;=\; m_2 \cdot p_{1\,lab}} \sigma_1^{tot}$$

$$\Rightarrow \quad \boxed{\mathrm{Im}\,\mathcal{M} = 2 E_{c.m}\, p_{c.m}\, \sigma^{tot}}$$

Pictorially:

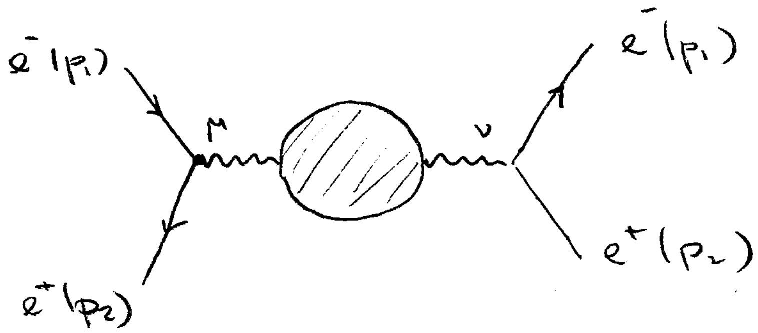


For e^+e^-, neglecting m_e, we have

$$\underline{\underline{\sigma_{e^+e^-} = \frac{1}{s}\,\mathrm{Im}\,\mathcal{M}(e^+e^- \to e^+e^-)}}$$

5.2. Operator analysis of e^+e^-

The diagrams for $e^+e^- \to e^+e^-$ whose imaginary part gives $e^+e^- \to$ hadrons have the form



The amplitude is

$$i\mathcal{M} = (-ie)^2 \; \bar{v}(p_2)\gamma^\mu u(p_1) \;\; \bar{u}(p_1)\gamma^\nu v(p_2)$$

$$\frac{-i}{s}\, i\Pi^h_{\mu\nu}(q)\, \frac{-i}{s}$$

$\Pi_{\mu\nu}$ is the hadronic part of the self energy and has the form

$$\Pi^{\mu\nu}_h(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\Pi_h(q^2)$$

[This follows from the Ward identity $q^\mu \Pi^{\mu\nu} = 0$

In our case only the $g^{\mu\nu}$-part contributes.

So $i\mathcal{M} = e^2 \frac{1}{s^2}\, i\Pi(q^2)\, \underline{\underline{s}}\, \bar{v}(p_2)\gamma_\mu u(p_1)\, \bar{u}(p_1)\gamma^\mu v(p_2)$

Now average over incoming spins, sum over outgoing

$$\frac{1}{4}\sum \bar{u}(p_1)\gamma^\mu v(p_2)\,\bar{v}(p_2)\gamma^\mu u(p_1)$$

$$= \frac{1}{4}\,\mathrm{tr}\left[\not{p}_1\gamma^\mu \not{p}_2 \gamma^\mu\right]$$

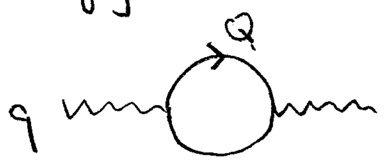
$$= \frac{1}{4}\Big(-\mathrm{tr}(\not{p}_1\not{p}_2 \underbrace{\gamma^\mu\gamma_\mu}_{4}) + 2p_{2\mu}\,\mathrm{tr}[\not{p}_1\gamma^\mu]$$

$$= \frac{1}{4}(-2)\,4\,p_1\cdot p_2 = -2s/2 = -s$$

$s=(p_1+p_2)^2 = 2p_1\cdot p_2$

$$\Rightarrow \quad \sigma(e^+e^-\to \text{hadrons}) = -\frac{4\pi\alpha}{s}\,\mathrm{Im}\,\Pi_h(s)$$

As a check, we should now evaluate the photon self-energy.



one finds $\mathrm{Im}\,\Pi(s+i\varepsilon) = -\frac{\alpha}{3}N_c\sum_q e_q^2$

$$\Rightarrow \quad \sigma(e^+e^-\to\text{hadrons}) = \frac{4\pi\alpha^{\textcircled{2}}}{3s}N_c\sum_q e_q^2 \quad ✓$$

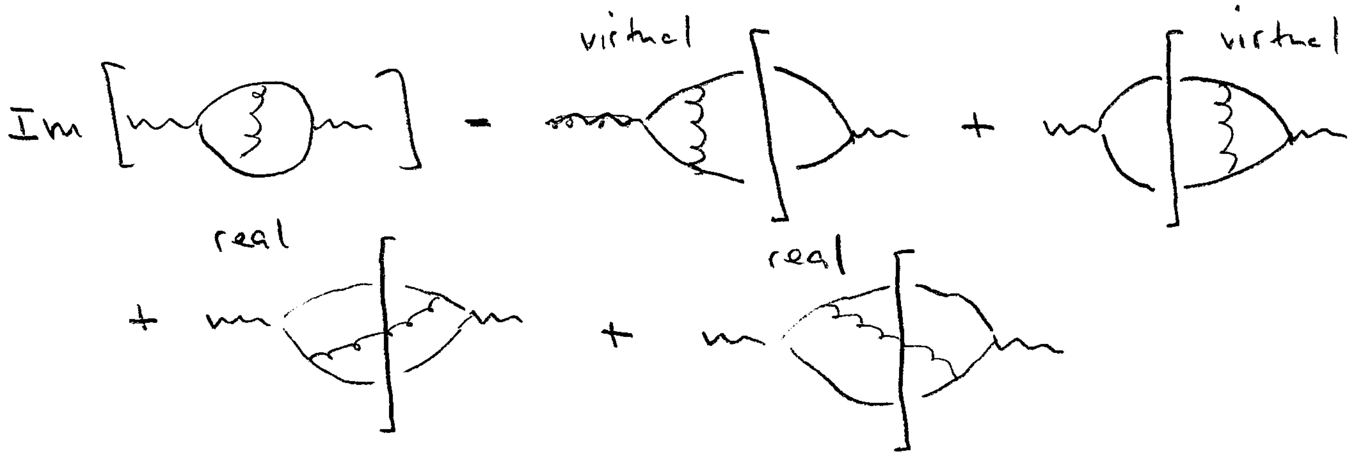
$$\to \Pi(s) = -\frac{\alpha}{3}N_c\sum_q e_q^2\left(-\frac{1}{\pi}\ln(-s-i\varepsilon)\right)$$

So we now get the cross section from a loop calculation alone.

At $O(\alpha_s)$

since off-shell Green's functions are IR finite, we have proven to all orders in α_s that the total cross section is finite!

The cancellation between real & virtual is manifest:



(The imaginary part arises when some particles in the loop go on the mass shell.)

The electromagnetic coupling of quarks has the form

$$J^\mu = \sum_q e_q \bar{q}_f \gamma^\mu q_f$$

so the self-energy is

$$i\Pi_h^{\mu\nu}(q) = (ie)^2 \int d^4x \, e^{iqx} \langle 0 | T\{ J^\mu(x) J^\nu(0) \} | 0 \rangle$$

In the limit $x \to 0$, we can expand the product of currents:

$$J_\mu(x) J_\nu(0) = C^{1}_{\mu\nu}(x) + C^{q\bar{q}}_{\mu\nu}(x) m_q \bar{q} q$$
$$+ C^{G^2}_{\mu\nu}(x) (G^a_{\mu\nu})^2$$

$$C^{1}_{\mu\nu} \sim (X^2)^{-3} \qquad C^{q\bar{q}}_{\mu\nu} \sim (x^2)^{-1} \qquad C^{G^2}_{\mu\nu} \sim (x^2)^{-1}$$

<u>If</u> the Fourier transform is dominated by $x^2 \approx 0$, we get

inserted by hand ..

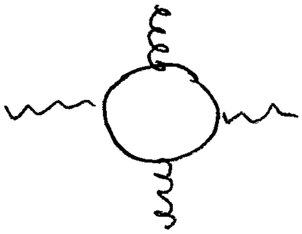
$$i\Pi_h^{\mu\nu}(q) = -ie^2\left(q^2 g^{\mu\nu} - q^\mu q^\nu\right)\Big[C^1(q^2)\,\mathbb{1} + C^{q\bar q}(q^2)\, m\bar q q + C^{G^2}(q^2)\,(G_{\mu\nu})^2\Big]$$

$$C^1 \sim (q^2)^0 \qquad \underbrace{C^{q\bar q} \sim \frac{1}{q^2} \qquad C^{G^2} \sim \frac{1}{q^2}}_{\text{suppressed at high energy}}$$

Since the OPE is an operator relation, we can take an arbitrary matrix element to obtain the Wilson coefficients. In particular, we can work with unphysical quark and gluon states.

$\longrightarrow \quad C^1(q^2)$

$\longrightarrow \quad C^{q\bar q}(q^2)$



$\longrightarrow \quad C^{G^2}(q^2)$

then, with the coefficients determined from perturbative calculations, we take the physical vaccum matrix element.

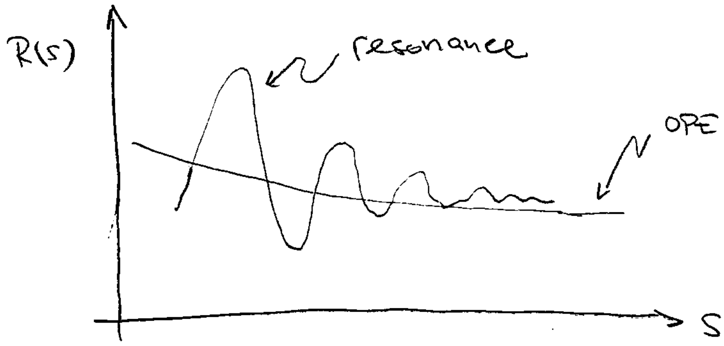
$$\Rightarrow \sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{s}\Big[\operatorname{Im} c^1(q^2) + \operatorname{Im} c^{q\bar{q}}(q^2) \langle 0 | \overbrace{m\bar{q}q}^{\sim (0{,}3\,\text{MeV})^4 \text{ quark condensate}} | 0 \rangle + \operatorname{Im} c^{G^2}(q^2) \underbrace{\langle 0 | (G^a_{\mu\nu})^2 | 0 \rangle}_{\substack{\text{Gluon condensate} \\ \sim (0{,}5\,\text{GeV})^4}}$$

We have separated the perturbative part (Wilson coefficient) from the nonperturbative part (operator matrix elements).

So, up to corrections of order Λ_{QCD}/q^2 from the condensates, the hadron cross section is equal to the partonic cross section from perturbation theory.

There is one potential worry concerning our result. The OPE is rigorous in Euclidean space, but we are using it in Minkowski space.

At sufficiently high energy we should be fine, but at lower energies, we can hit a QCD resonance, and the OPE will break down



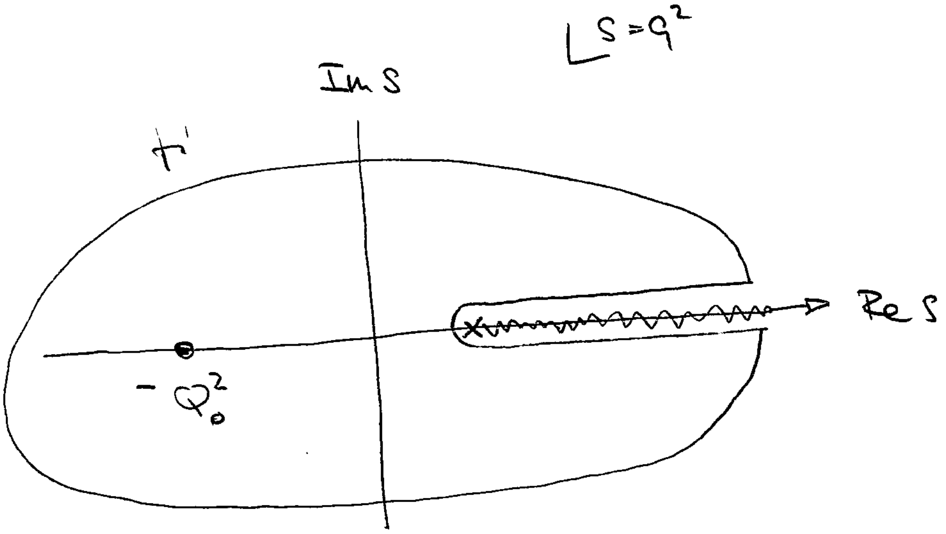
The reason is that near the resonance $R(s) \sim \frac{1}{s - M_{res}^2 + i M_{res}\Gamma}$ the expansion in $\frac{1}{s}$ breaks down.

It turns out that one can avoid calculating $\Pi(q^2)$ in the Minkowski region and at the same time make use of all available data.

The magic of complex analysis!

If we look at $\Pi(q^2)$ in the complex plane, we find that it is analytic everywhere, except for a cut from $s = (2m_\pi)^2 \ldots \infty$.

[In perturbation theory, the cut starts at $s = (2m_q)^2$.]



The integral $\frac{1}{2\pi i}\oint_\Gamma ds\, \Pi(s) = 0$ vanishes.

Now consider

$$I_n = -4\pi\alpha \oint_\Gamma \frac{ds}{2\pi i} \frac{1}{(q^2+Q_0^2)^{n+1}} \Pi(s)$$

$$= -4\pi\alpha \frac{1}{n!} \frac{d^n}{ds^n} \Pi(s) \Big|_{s=-Q_0^2}$$

At $s = -Q_0^2$ it is safe to evaluate $\Pi(s)$ since we are away from all singularities at Euclidean values of s.

There is another way to evaluate the integral

$$I_n = -4\pi\alpha \int_T \frac{ds}{2\pi i} \frac{1}{(s+Q_0^2)^{n+1}} \Pi(s)$$

$$= -4\pi\alpha \int_0^\infty \frac{ds}{2\pi} \frac{1}{(s+Q_0^2)^{n+1}} \frac{1}{i}\left[\Pi(s+i\varepsilon) - \Pi(s-i\varepsilon)\right]$$

$$= -4\pi\alpha \int_0^\infty \frac{ds}{2\pi} \frac{1}{(s+Q_0^2)^{n+1}} 2\,\mathrm{Im}\,\Pi(s)$$

$$= \frac{1}{\pi}\int_0^\infty ds \frac{s}{(s+Q_0^2)^{n+1}} \sigma(s)$$

So $\Pi(s)$ at Euclidean momentum is equal to an integral over the cross section.

Now evaluate

$$\underbrace{-4\pi\alpha \frac{1}{n!}\frac{d^n}{ds^n}\Pi(s)\Big|_{s=-Q_0^2}}_{\text{using the OPE.}} = \underbrace{\frac{1}{\pi}\int_0^\infty ds \frac{s}{(s+Q_0^2)^{n+1}}\sigma(s)}_{\text{using data}}$$

In this way one gets one of the most precise determinations of $\alpha_s(\mu)$ and of the heavy quark masses m_c and m_b.

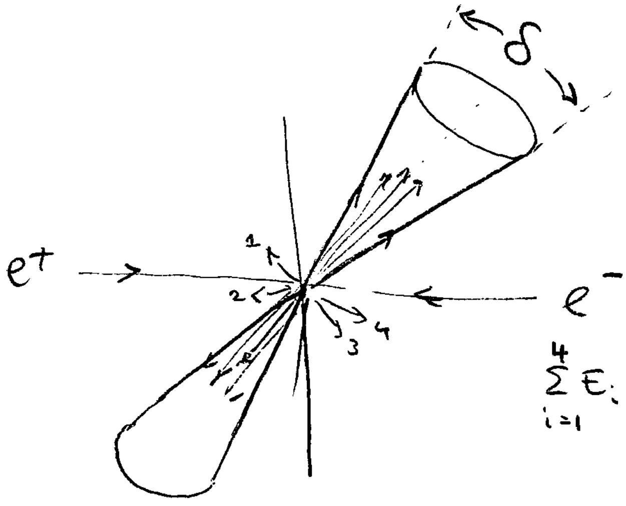
The above relation is called a sum rule, or dispersion relation.

6. Event-shape variables and jets

We have seen that the total $e^+e^- \rightarrow$ hadrons cross section can be calculated in perturbation theory. We now want to consider more complicated observables. A minimal requirement for sensible observables is that the infrared divergences present in individual diagrams cancel. These arise from soft and collinear partons (quarks and gluons).

The first definition which fulfills this requirement was given by Sterman & Weinberg '77.

The cross section for two Sterman-Weinberg jets is defined as follows:

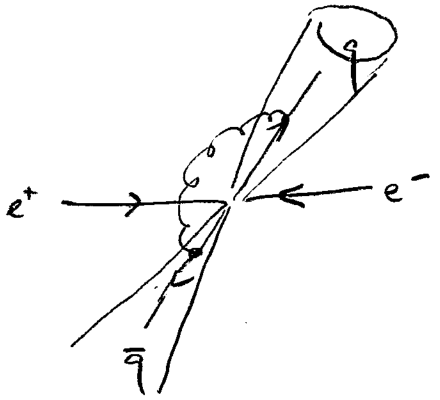


$$\sum_{i=1}^{4} E_i < \varepsilon E$$

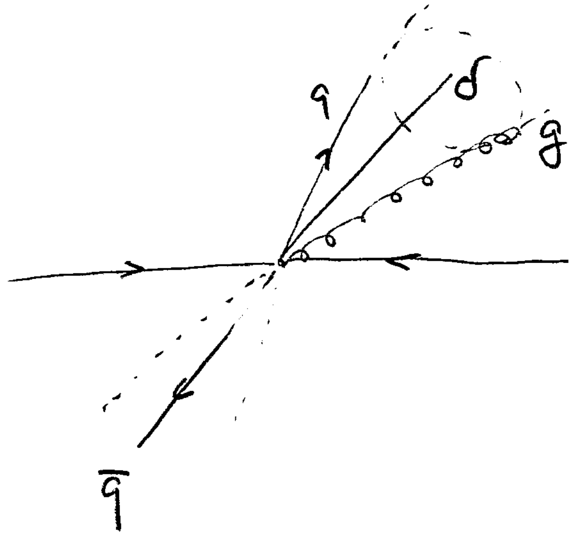
An event contributes if all the energy of its particles is contained in two cones of opening angles δ, except for a fraction ε of the total energy, so $\sigma_{SW} \equiv \sigma_{SW}(\delta, \varepsilon)$.

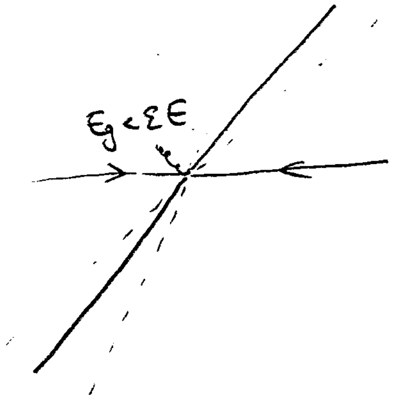
Let's see how this works for $e^+e^- \to \bar{q}q$ and $e^+e^- \to \bar{q}qg$

1.) All of the Born-level (lowest order) and all of the virtual corrections contribute, irrespective of δ and ε



2.) The real emission contributes, if the gluon energy is small $E_g < \varepsilon E$ or if the angle between the quark and gluon is smaller than δ.





So the Sterman Weinberg cross section includes <u>all</u> IR singular parts of the cross section.

Since the singularities cancelled for the total cross section, also σ_{SW} is finite.

The crucial feature of σ_{SW} is that it remains unchanged by very soft or very collinear emissions. The insensitivity to these low energy contributions should also guarantee that the results are not strongly affected by nonperturbative effects.

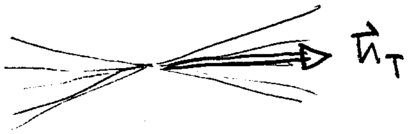
Many IR safe observables have been defined after the work of Sterman and Weinberg. Broadly speaking, they fall into two classes: event shapes and jet algorithms.

6.1. Event-shape variables

The classic shape variable is thrust defined as

$$T = \frac{1}{Q} \max_{\{\hat{n}\}} \sum_i |\vec{p}_i \cdot \hat{n}| \qquad Q = \sum_i |p_i|$$

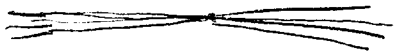
The vector which maximizes T, $\hat{n}_T$ is the thrust axis, the direction of maximum momentum transfer,



(or the opposite direction)

Let's look at two extreme cases

1.) All particles have momenta in the same direction $\vec{p}_i = \pm |\vec{p}_i| \, \hat{n}_T$



"pencil-like"

$$T = \frac{1}{Q} \sum_i |\vec{p}_i| = 1$$

2.) The event is completely spherical. In this case any direction serves as the thrust axis. Let's chose $\hat{n}_T = (0,0,1)$.

We now have particles in any direction, all with the same momentum $|\vec{p}|$

So $Q = \sum_i |p_i| = |\vec{p}| \int d\Omega = 4\pi |\vec{p}|$

$$T = \frac{1}{4\pi|\vec{p}|} \int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\varphi \; |\vec{p}|\,|\cos\theta|$$

$$= \frac{1}{2} 2\int_0^1 d\cos\theta \; \cos\theta = \underline{\underline{\frac{1}{2}}}$$

This is the minimum value of thrust.

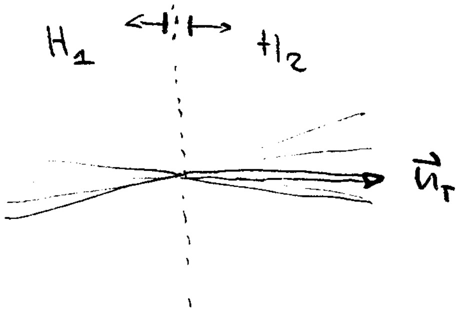
[Exercise: What is the minimum value with 3 particles?]

Thrust is infrared safe, since it is unchanged by soft and collinear emissions.

Let's consider a collinear splitting, $\vec{p} \to \vec{p}_A + \vec{p}_B$ with $\vec{p}_A \parallel \vec{p}_B$:

$$|\vec{p}\cdot\vec{n}_T| = |\vec{p}_A\cdot\vec{n}_T| + |\vec{p}_B\cdot\vec{n}_T| \quad \checkmark$$

With the thrust axis at hand, one can define several other shape variables. Define hemispheres:



* Hemisphere mass

$$M^2_{H_k} = \Big(\sum_{i \in H_k} |\vec{p}_i|\Big)^2 \qquad k = 1,2.$$

Heavy jet mass $\rho_H = \frac{1}{Q^2} \max\left(M^2_{H_1}, M^2_{H_2}\right)$

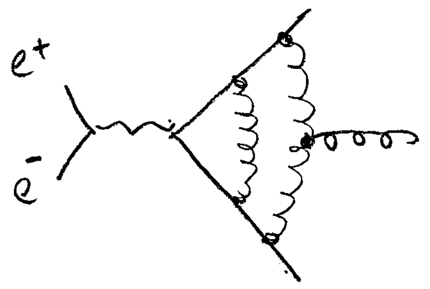
Light jet mass $\rho_L = \frac{1}{Q^2} \min\left(M^2_{H_1}, M^2_{H_2}\right)$

* Broadening: $B_{1,2} = \frac{1}{Q^2} \sum_{i \in H_{1,2}} |\underbrace{\vec{n}_T \times \vec{p}_i}_{\text{momentum transverse to the thrust axis.}}|$

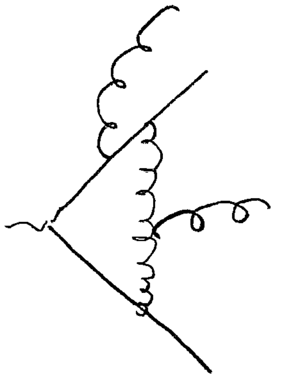
Total broadening $B_T = B_1 + B_2$

Wide broadening $B_W = \max(B_1, B_2)$

These event shapes have recently* been calculated at NNLO. The relevant diagrams are

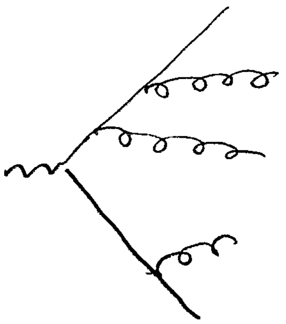


$\sigma_{q\bar{q}g}$ at 2 loops



σ_{qqgg} at 1 loop

\+ many others



σ_{qqggg} at tree level

$\vdots$

---

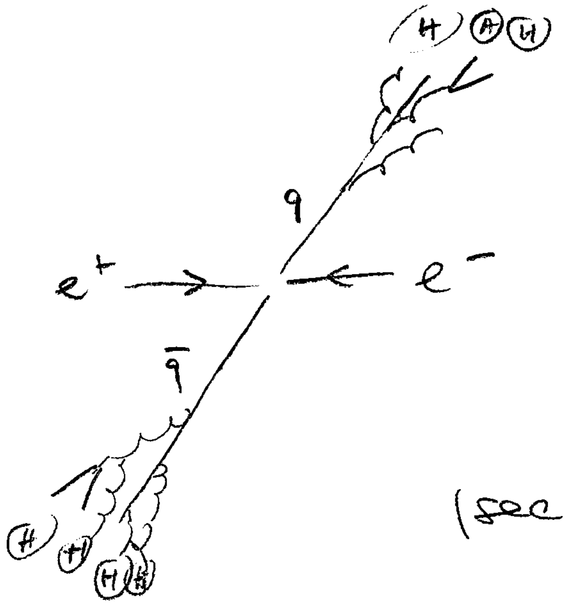
* Gehrmann ~~De~~Ridder, Gehrmann, Glover, Heinrich '07
Weinzierl '08

Dealing with the IR div's which appear in the loop and phase-space integrals (and cancel in the sum of all contributions) is very nontrivial.

Comparing the calculated and measured distribution allows one to extract a value of α_s. (see figures).

6.2 Jet algorithms

A disadvantage of event shapes is that they are not closely related to the underlying perturbative QCD dynamics. Collider events often have the form of jets, collimated sprays of hadrons which result after soft & collinear emissions and hadronisation



(see figures)

Jet algorithms identify such jets. In order for the jet rates to be perturbatively calculable, the algorithms should be IR safe.

There are two categories of jet algorithms

A.) <u>Cone-type</u> (Sterman-Weinberg, JetClu, Midpoint, ..., SISCone)

Cluster according to distance in coordinate space.

Put cones around dominant directions of energy flow.

B.) <u>Sequential</u> (k_T-algorithm, JADE, Cambridge/Aachen, ...)

Cluster according to distance in momentum space.

"Undo" the branchings in the perturbative evolution.

A thorough discussion of many algorithms can be found in 0906.1833 by Gavin Salam.

6.2.1 Sequential algorithms

JADE algorithm

1.) For each pair of particles calculate

$$y_{ij} = \frac{2E_iE_j(1-\cos\theta_{ij})}{Q} \left[= \frac{(p_i-p_j)^2}{Q} \text{ if } p_i^2=p_j^2=0 \right]$$

2.) Find minimum y_{min} of the y_{ij}'s

3.) If $y_{min} < y_{cut}$, then recombine i and j into new "particle" (or "pseudojet") and repeat from 1.)

4.) otherwise, declare all remaining particles to be jets and terminate.

The larger one chooses y_{cut}, the fewer jets one gets. The algorithm is infrared safe, because soft and collinear particles get clustered together immediately.

Disadvantage: soft particles get clustered together even if they move in opposite directions.

k_T - algorithm in e^+e^-

Identical to JADE, except

$$y_{ij} = \frac{2\min(E_i^2, E_j^2)(1-\cos\theta_{ij})}{Q^2}$$

For $\theta_{ij} \to 0$ $\quad y_{ij} \to \min(E_i^2, E_j^2)\theta_{ij}^2$ .. transverse momentum

The use of the minimal energy ensures that soft particles are clustered with larger particles in similar directions.

k_T-, anti-k_T-, and C/A (Cambridge/Aachen) algorithm

At hadron colliders, the value of Q is not known, because the colliding quarks & gluons only carry a fraction of the proton momentum. For the same reason, the collisions are not happening in the c.m.s. Because of this one uses

Rapidity $\quad y = \frac{1}{2} \ln \left(\frac{E + p_z}{E - p_z} \right) \quad [\text{or } \eta = -\ln \tan \theta/2]$

transverse momentum $\quad p_t = \sqrt{p_x^2 + p_y^2} \quad$ (beam in the z-dir)

azimuthal angle $\quad \phi = \arctan \frac{p_y}{p_x}$

as variables.

Rapidity is useful since rapidity differences are independent of boosts along the beam direction.

The distance measure in the above algoriths is

$$d_{ij} = \min(p_{t_i}^{2p}, p_{t_j}^{2p}) \frac{\Delta R_{ij}}{R^2}$$

$$d_{iB} = p_{ti}^{2p} \quad ; \quad \Delta R_{ij} = \sqrt{(y_i - y_j)^2 + (\phi_i - \phi_j)^2}$$

$\nwarrow$ "distance to beam"

$p=1$: k_t -algorithm (Catani et al, Ellis and Soper '93)

$p=0$: C/A (Wobisch, Wengler '99)

$p=-1$: anti-k_t (Cacciari, Salam and Soyez '08)

The clustering is ~~pu~~ performed as follows

1.) Calculate all d_{iB}'s and d_{ij}'s and find the minimum.

2.) If it is a d_{ij}, combine i & j, goto 1.)

3.) If it is a d_{iB} declare it to be a jet and remove it from the list, goto 1.

4.) Stop when no particles remain.

Note that all particles are now part of a jet, since there is no d_{cut}

k_t was advocated by theorists because it is IR safe. It was disliked (and hardly used) by experimentalists because it produced irregular jets and was computationally expensive for many particles.

6.14.

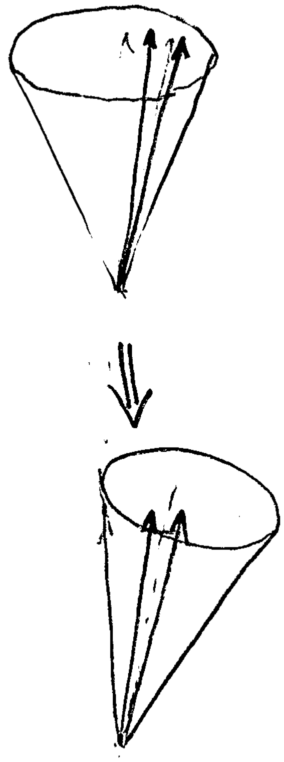
The irregular jets arise from combining soft particles early because their d_{ij}'s are low. This is improved by anti-k_t which combines hard particles first. It leads to regular jet-shapes, similar to cone algorithms and will presumably be the main algorithm used at the LHC.

6.2.2. Cone algorithms

We have already seen one example of a cone algorithm, the Sterman-Weinberg cross section.

Experimenters prefer cone algorithms, since they have a more immediate physical interpretation and are computationally less expensive (at least if one compromises on IR safety ...)

Modern cone algorithms are iterative. One places a cone, calculates the total momentum of all particles inside the cone. Then one recenters the cone around this direction and repeats until one has a stable cone.



Two problems:

i.) What should be taken as the starting cones, the seeds of the iteration?

ii.) What should be done if two cones overlap?

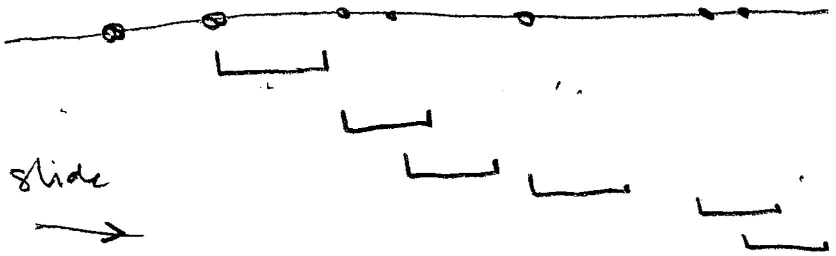
The solution to i.) is to use a <u>seedless</u> cone algorithm: take all subsets of particles, calculate the total momentum, use it as the cone axis and see whether it corresponds to a stable cone. Since there are 2^N subsets, this is not feasible for a large number of particles.

$\Rightarrow$ Experimentalists used seeds, which leads to IR unsafety, since additional soft particles can lead to new stable cones.

In '07 Salam and Soyez came up with a polynomial time seedless cone algorithm (SISCone)

Let's explain the procedure in 1-d:

A cone corresponds to a line of length ℓ. To find all stable cones one can take all subsets of points and check whether they are within a distance ℓ. This is *very* inefficient. A better algorithm is to slide a segment of length ℓ along the line



In 2-d: (ϕ, y) one moves around a circle (see figures).

A solution to ii) is to use a so-called split-merge procedure

1.) Find the stable cone with the largest p_t, call it a.

2.) Find the next largest cone that shares particles, label it b. If no such cone exists, remove cone a and its particles, add a to the final list of jets.

3.) Calculate the p_t of the shared particles $p_{t,\text{shared}}$

a.) If $p_{t,\text{shared}} / p_{t,b} > f$, replace a & b with a single cone containing all particles

b.) otherwise split the two by assigning the shared particles to one of the jets.

4.) Repeat 1, until no cones are left.

6.19.

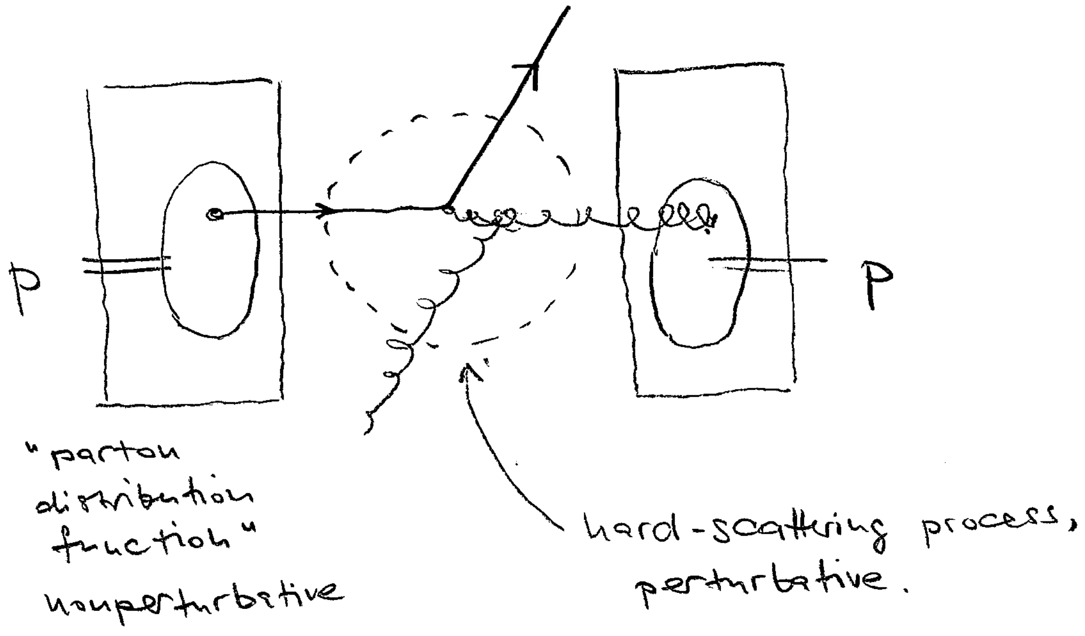
The program FastJet implements many algorithms. One can feed it a list of particle momenta, choose an algorithm and it spits out the jets.

7. Soft-Collinear Effective Theory

We have defined observables such as event shapes and jet rates which are insenstive to low energy QCD effects from soft and collinear emissions.

However in hadronic collisions, low energy QCD effects are unavoidable: the scattering process involves a bound state of quarks and gluons, which cannot be treated perturbatively.

What saves the day are factorization theorems.



The parton distribution functions (PDFs) give the probability* to find a quark or gluon with momentum $x \cdot P$ inside a hadron with momentum P.

The cross section factorizes as

$$\sigma^{had}(P_1,P_2) = \underset{\substack{\uparrow \\ \{a,b\} = \{g,u,d,s,\ldots \\ \bar{u},\bar{d},\bar{s},\ldots\}}}{\sum_{a,b}} \int_0^1 dx_1 \int_0^1 dx_2 \, \underbrace{f_{a/H_1}(x_1)\, f_{b/H_2}(x_2)}_{\text{PDFs}} \, \underbrace{\sigma^{ab}(x_1 P_1, x_2 P_2)}_{\substack{\text{partonic} \\ \text{cross-section}}}$$

The proof of such factorization theorems is quite involved, so most (all?) books avoid the subject. Using modern effective theory methods makes such proofs much simpler and more transparent. In the following we'll develop the necessary formalism. It will take some effort to construct SCET, but once we're done the factorization proofs will be relatively straightforward.

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*.) Roughly speaking. They are renormalized which makes the interpretation less straightforward.

7.1. Strategy of regions

The "strategy of regions" (Beneke and Smirnov '97) is a very general and efficient method to expand loop integrals around various limits. The expansion is obtained by splitting the integral into contributions from different regions. In our case, the regions will be regions of soft and collinear momentum, but let's first warm up with a 1-d example integral

$$I = \int_0^\infty dk \frac{k}{(k^2+m^2)(k^2+M^2)} = \frac{\ln\left(\frac{M}{m}\right)}{M^2-m^2}$$

$$= \ln\left(\frac{M}{m}\right)\frac{1}{M^2}\left\{1+\frac{m^2}{M^2}+\frac{m^4}{M^4}+\dots\right\} \quad \text{for } m^2 \ll M^2$$

Can we expand on the level of the integrand? No:

$$I \overset{?}{=} \int_0^\infty dk \frac{k}{(k^2+M^2)}\,\frac{1}{k^2}\left(1-\frac{m^2}{k^2}+\dots\right)$$

IR - divergence!

Problem: In the region $k \sim m$, the expansion of the integrand is not justified!

Solution: Split the integration in two regions

$$m << \Lambda << M.$$

$$I = \Big(\underbrace{\int_0^\Lambda dk}_{(I)} + \underbrace{\int_\Lambda^\infty dk}_{(II)} \Big) \frac{k}{(k^2+m^2)(k^2+M^2)}$$

in (I) $k \sim m << M$, expand

$$\frac{1}{(k^2+m^2)(k^2+M^2)} = \frac{1}{(k^2+m^2)} \frac{1}{M^2} \left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \dots \right)$$

in (II) $m << k \sim M$, expand

$$\frac{1}{(k^2+M^2)} \frac{1}{k^2} \left(1 - \frac{m^2}{k^2} + \frac{m^4}{k^4} - \dots \right)$$

$$I_{(I)} = - \frac{\Lambda^2}{2M^4} - \frac{1}{M^2} \ln\left(\frac{m}{\Lambda}\right) + \dots$$

$$I_{(II)} = + \frac{\Lambda^2}{2M^4} - \frac{1}{M^2} \ln\left(\frac{\Lambda}{M}\right) + \dots$$

In the sum

$$I = I_{(I)} + I_{(II)} = - \frac{1}{M^2} \ln\left(\frac{m}{M}\right) \checkmark$$

the dependence on the separator Λ cancels, and we indeed reproduce the first term in the expansion of the integral.

Note that there is a close correspondance between what we did and the concept of an effective theory. We have split the integral in a low energy region (I) and a high energy region (II). Λ is the the UV cut-off in the low energy region.

While our method works fine, it is very hard to evaluate loop integrals in cut-off regularization. It turns out that we can get the same result using dimensional regularization instead of a hard cut-off.

Consider the integral in dimensional regularization

$$I = \int dk\, k^{-\varepsilon} \frac{k}{(k^2+m^2)(k^2+M^2)}$$

⌈ This simplified version is good enough; our integral is finite and we only want $\varepsilon \to 0$ at the end of the day. ⌋

Now calculate the contributions of (I) & (II), but without a cut-off.

$$I_{(I)} = \int_0^\infty dk \frac{k^{-\varepsilon}}{(k^2+m^2)} \frac{1}{M^2} \left(1 - \frac{k^2}{M^2} + \frac{k^4}{M^4} + \dots\right)$$

$$= + \frac{1}{M^2}\left[+\frac{1}{\varepsilon} - \ln(m)\right]$$

↖ UV divergence "$\varepsilon > 0$"

$$I_{(II)} = \int_0^\infty dk \frac{k^{-\varepsilon}}{(k^2+M^2)} \frac{1}{k^2} \left(1 - \frac{n^2}{k^2} + \dots\right)$$

$$= + \frac{1}{M^2}\left[-\frac{1}{\varepsilon} + \ln(M)\right]$$

↖ IR divergence "$\varepsilon < 0$"

The sum is

$$I = I_{(I)} + I_{(II)} = -\frac{1}{M^2}\ln\left(\frac{m}{M}\right) \checkmark$$

The $\frac{1}{\varepsilon}$ divergences have cancelled.

That the procedure works is surprising at first sight. It looks like we are double counting by integrating over the full phase-space in both (I) & (II).

One way to see that this is not the case is to subtract from our low energy integral it's expansion around high energies

$$I'_{(I)} = \int_0^\infty dk \, \frac{k^{-\varepsilon}}{M^2}\left[\frac{1}{k^2+m^2} - \left(\frac{1}{k^2} - \frac{m^2}{k^4} + \frac{m^4}{k^6}\right)\right]$$

After the subtraction, the integrand $I'_{(1)}$ is no longer sensitive to the region of large k, since the integrand goes like $\frac{1}{k^8}$ for large k.

However, all subtraction terms are scaleless and vanish in dim. reg.

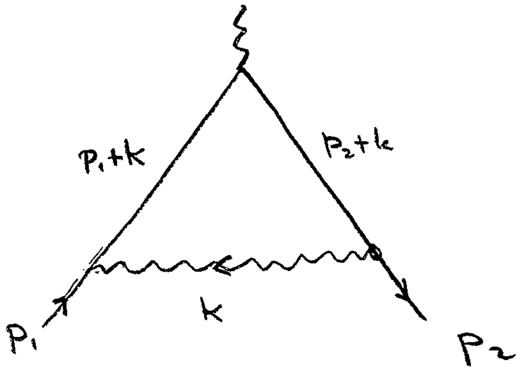
So $I'_{(I)} = I_{(I)}$.

$\rightsquigarrow$ The overlap between regions corresponds to scaleless integrals.

For applications, such as the expansion of Euclidean integrals in small or large masses, it was proven that the above procedure leads to the correct result also for n-loop integrals. Such a proof is however still missing for the application we will now consider.

7.1.1. Sudakov problem

Let us now look at the expansion in a situation where particles have large energy but small invariant masses p^2 and expand in this limit. The simplest example is the integral



$$p_1^2 \sim p_2^2 \ll (p_1-p_2)^2 = s$$

For simplicity, let's just consider the scalar integral

$$\int d^dk \frac{1}{(p_1+k)^2-i\varepsilon}\,\frac{1}{(p_2+k)^2-i\varepsilon}\,\frac{1}{k^2-i\varepsilon} = -i\pi^{d/2}\overbrace{\frac{e^{-\gamma_E\varepsilon}}{s}}^{C} V(p_1^2,p_2^2,s)$$

To perform the expansion, it is useful to introduce light-like reference vectors in the directions of the particles.

7.10.

Assume that the incoming particle flys in the $+z$ direction, and p_2 is in the $-z$ direction.

Introduce $n^\mu = (1, 0, 0, 1)$

$$\bar{n}^\mu = (1, 0, 0, -1)$$

$$n^2 = \bar{n}^2 = 0 . \quad n \cdot \bar{n} = 2 .$$

Any four-vector p^μ can be written as

$$p^\mu = (n \cdot p) \frac{\bar{n}^\mu}{2} + (\bar{n} \cdot p) \frac{n^\mu}{2} + p_\perp^\mu$$

$$= p_+^\mu + p_-^\mu + p_\perp^\mu .$$

Where $p_\perp \cdot n = p_\perp \cdot \bar{n} = 0.$

Note that $p^2 = 2 \cdot n \cdot p \, \bar{n} p \, \frac{n \cdot \bar{n}}{4} + p_\perp^2$

$$= n \cdot p \, \bar{n} p + p_\perp^2 .$$

Since p_1 flys in the z-direction $\bar{n} \cdot p_1$ is large, but since p_1^2 is small $p_\perp^\mu$ and $n \cdot p$ must be small.

More precisely, the components scale as

$$\qquad\qquad (n\cdot p\,,\ \bar{n}\cdot p\,,\ p_\perp)$$
$$p_1^\mu \sim E\,(\lambda^2\,,\ 1\,,\ \lambda) \qquad \text{with } \lambda^2 \sim \frac{p^2}{E^2}$$

$$p_2^\mu \sim E\,(1\,,\ \lambda^2\,,\ \lambda)$$

To expand the loop integral we need to consider the following regions of the loop-momentum k^μ:

hard: $k^\mu \sim (1\,,\ 1\,,\ 1)$

1-collinear: $k^\mu \sim p_1^\mu \sim (\lambda^2\,,\ 1\,,\ \lambda)$

2-collinear: $k^\mu \sim p_2^\mu \sim (1\,,\ \lambda^2\,,\ \lambda)$

soft*: $k^\mu \sim (\lambda^2\,,\ \lambda^2\,,\ \lambda^2)$

Note: any other scaling $k^\mu \sim (\lambda^a\,,\ \lambda^b\,,\ \lambda^c)$ leads to scaleless integrals after expanding.

---

* This is often called ultra-soft to distinguish it from $(\lambda, \lambda, \lambda)$.

7.12.

So let's look at the contributions from the different regions:

hard:

$$\int d^dk \frac{1}{(2p_1\cdot k + k^2)(2p_2\cdot k + k^2)\,k^2} + \dots = +C\cdot V_h$$

$$V_h = \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln(-s) + \frac{1}{2}\ln^2(s) - \frac{\pi^2}{12}$$

1-collinear

$$\int d^dk \frac{1}{(p_1+k)^2}\,\frac{1}{\underbrace{(2p_{2+}\cdot k + i\varepsilon)}_{O(\lambda^0)}}\,\frac{1}{k^2} = +C\cdot V_{c_1}$$

$$V_{e_1} = -\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon}\ln(-p_1^2) - \frac{1}{2}\ln^2(-p_1^2) + \frac{\pi^2}{12}$$

soft:

$$\int d^dk \frac{1}{(2p_{\perp}\cdot k_+ + p_1^2)(2p_{2+}\cdot k_- + p_2^2)\,k^2} = C\cdot V_s$$

$$= \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon}\ln\left(\frac{(-p_1^2)(-p_2^2)}{s}\right) + \frac{1}{8}\ln^2\left(\frac{(-p_1^2)(-p_2^2)}{s}\right) + \frac{\pi^2}{4}$$

In the sum of all terms the divergences cancel and one obtains:

$$V = V_h + V_{c_1} + V_{c_2} + V_s = \underline{\frac{1}{4} \ln^2\left(\frac{(-p_1^2)(-p_2^2)}{s^2}\right)}$$

Let's calculate one of these integrals explicitly:

$$c\, V_s = \int d^q k \frac{1}{(2p_{1-}\cdot k_+ + p_1^2)(2p_{2+}\cdot k_- + p_2^2)\, k^2}$$

$$\left[\int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \frac{2}{(a+b\eta_1+c\eta_2)^3} = \frac{1}{abc} \right.$$

$$= \int d^q k \int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \frac{2}{\left[(k+\eta_1 p_{1-}+\eta_2 p_{2+})^2 - (\eta_1 p_{1-} + p_{2+}\cdot\eta_2)^2 - \eta_1 p_1^2 - \eta_2 p_2^2\right]^3}$$

$$= -i\pi^{d/2} \frac{\Gamma(3-d/2)}{2} \cdot 2 \int_0^\infty d\eta_1 \int_0^\infty d\eta_2 \left[\eta_1 p_1^2 + \eta_2 p_2^2 + 2\eta_1\eta_2 \overbrace{p_{1-}\cdot p_{2+}}^{\approx s}\right]^{d/2-3}$$

$$= -i\pi^{d/2}\,\Gamma\left(3-\frac{d}{2}\right)\frac{1}{p_1^2 p_2^2}\int_0^\infty d\eta_1 \int_0^\infty d\eta_2\,\left[\eta_1+\eta_2+\eta_1\eta_2\cdot a\right]^{-1-\varepsilon}$$

with $a = \dfrac{s}{(-p_1^2)(-p_2^2)}$

$$\eta_1^{-1-\varepsilon}\left[1+\eta_2+\eta_1\eta_2\cdot a\right]^{-1-\varepsilon}$$

$$= -i\pi^{d/2}\,\Gamma(1+\varepsilon)\frac{1}{p_1^2 p_2^2}\int_0^\infty d\eta_1 \int_0^\infty d\eta_2\,\eta_1\left[\eta_1+\eta_1\eta_2+ -\eta_1^2\eta_2\cdot a\right]^{-1-\varepsilon}$$

$$= -i\pi^{d/2}\,\Gamma(1+\varepsilon)\frac{1}{p_1^2 p_2^2}\underbrace{\int_0^\infty d\eta_1\,\frac{\eta^{-1-\varepsilon}}{1+\eta_1}}_{\Gamma(1-\varepsilon)\Gamma(\varepsilon)}\underbrace{\int_0^\infty d\eta_2\,\left[1+\eta_2\right]^{-1-\varepsilon}}_{1/\varepsilon}$$

$$= -i\pi^{d/2}\,\Gamma(1-\varepsilon)\,\Gamma(\varepsilon)^2)\,\frac{1}{s}\left(\frac{(-p_1^2)(-p_2)^2}{s}\right)^{-\varepsilon}$$

$$\leadsto V_h = \left(\frac{1}{\varepsilon^2}+\frac{\pi^2}{4}\right)\left(\frac{(-p_1^2)(-p_1^2)}{s}\right)^{-\varepsilon}$$

7.2. Scalar SCET

In the last lecture, we have expanded a triangle integral around the limit where the in- and outgoing particles were energetic $p_1^2 \ll E_1^2$, $p_2^2 \ll E_2^2$ [or, to say it in a Lorentz invariant, way $p_1^2 \sim p_2^2 \ll (p_1-p_2)^2$]

We will now construct an effective theory, which implements this expansion on the level of the Lagrangian. Calculating with the leading-order Lagrangian, one automatically obtains the leading order expansion of the full-theory results. Adding also subleading terms, one can systematically also obtain higher-order corrections.

The structure of the effective theory relates very closely to the expansion in the strategy of regions method. The correspondence is basically one-to-one: the Feynman rules in Soft-Collinear Effective Theory simply produce the contributions of the different regions.

Let us briefly recapitulate what the ingredients of the strategy of region were:

- Light-cone vectors in the directions of the energetic particles $n^\mu = (1,0,0,1)$, $\bar{n}^\mu = (1,0,0,-1)$
- Expansion parameter $\lambda^2 \sim \frac{p^2}{E^2}$
- Momentum regions

	$(n\cdot k\,,\bar{n}\cdot k\,,k_\perp)$	
hard	$(1\,,1\,,1)$	$k^2 \sim 1$
coll-1	$(\lambda^2\,,1\,,\lambda)$	$k^2 \sim \lambda^2$
coll-2	$(1\,,\lambda^2\,,\lambda)$	$k^2 \sim \lambda^2$
soft	$(\lambda^2\,,\lambda^2\,,\lambda^2)$	$k^2 \sim \lambda^4$

- Expand in each region, integrate over all momenta in each case.

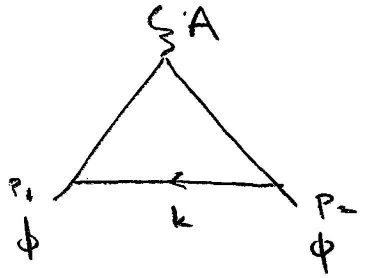
For simplicity, we'll first construct the effective theory for ϕ^3-theory: the construction is basically the same as for QCD.

The full theory Lagrangian is

External current

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{g}{3!}\phi^3 + A\phi^2 .$$

Up to a prefactor, our scalar integral corresponds to



To obtain the effective theory, we replace

$$\phi(x) \longrightarrow \phi_s(x) + \phi_{c_1}(x) + \phi_{c_2}(x)$$

soft field   collinear fields

We do not introduce a hard field: we'll integrate out the hard contribution and absorb it into the Wilson coefficients (i.e. the couplings) of operators built from soft and collinear fields, in analogy to the OPE.

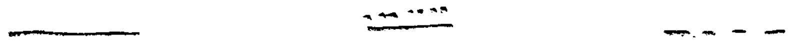
Splitting the fields into three modes is not good enough yet: to get the strategy of region result, we have to expand in the small momentum components.

How does the collinear field $\phi_{c_1}(x)$ know that that its momentum scales as $(\lambda^2, 1, \lambda)$?

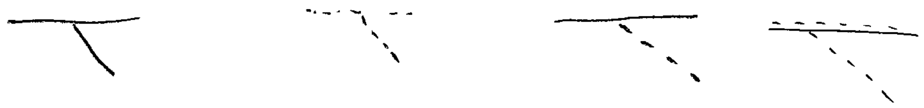
To have this scaling, we

- introduce external sources which inject the appropriate momentum.
- only allow interactions compatible with the scaling.

Expanding $\phi \to \phi_{c_1} + \phi_{c_2} + \phi_s$, we keep only the following terms:

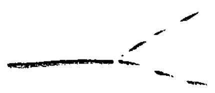


$$\mathcal{L}_{SCET} = \frac{1}{2}(\partial_\mu \phi_{c_1})^2 + \frac{1}{2}(\partial_\mu \phi_{c_2})^2 + \frac{1}{2}(\partial_\mu \phi_s)^2 - \frac{g}{3!}(\phi_{c_1}^3 + \phi_{c_2}^3 + \phi_s^3) - \frac{g}{2!}(\phi_{c_1}^2 \phi_s + \phi_{c_2}^2 \phi_s)$$



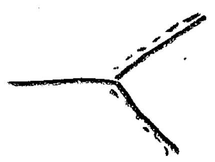
Terms forbidden by momentum conservation

$\phi_{c_1}\phi_s^2$



An energetic particle decays to two particles with small energ

$\phi_{c_1}\phi_{c_2}^2$



A particle moving fast in the +z direction turns into two moving in the −z direction.

For the last two terms we need to implement the expansion in small momentum. In position space, this corresponds to an expansion in derivatives. To obtain the expansion, write the $\phi_{c_1}^2 \phi_s$ term in terms of Fourier transformed fields.

$$\delta L_{int} = \int d^4x \, \phi_{c_1}^2(x) \, \phi(x) =$$

$$\int d^4x \int_{p_1, p_2, p_3} e^{-i(p_1+p_2+p_3)x} \, \phi_{c_1}(p_1) \phi_{c_2}(p_2) \phi_s(p_3)$$

$$\begin{array}{lllll}
 & & n\cdot p & \bar{n}\cdot p & p_\perp \\
(p_1+p_2+p_3)^\mu & \text{scales} & (\lambda^2, & 1, & \lambda) \\
x^\lambda & \text{scales} & (1, & 1/\lambda^2, & 1/\lambda) \\
p_s & \text{scales} & (\lambda^2, & \lambda^2, & \lambda^2) \\
 & & p_+ + & p_- + & p_\perp
\end{array}$$

$$\Rightarrow \quad \underset{}{p_s \cdot x} = \underset{O(1)}{p_{s+} x_-} + \underset{O(\lambda)}{p_{s\perp} \cdot x_\perp} + \underset{O(\lambda^2)}{p_{s-} \cdot x_+}$$

$$\delta L_{int} = \int d^4x \, \phi_{c_1}^2(x) \, \phi(x)$$

$$= \int d^4x \, \phi_{c_1}^2(x) \left[1 + x_\perp \cdot \partial_\perp + x_+ \cdot \partial_{x_-} + \dots \right] \phi_s(x) \Big|_{x=x_-}$$

$$= \int d^4x \, \phi_{c_1}^2(x) \, \phi_s(x_-) + \dots$$

The expansion in derivatives is called "multipole-expansion" in the original references (Beneke & Feldmann et al. '02,'03)

There is an alternative formulation, the "Label-formalism" by Bauer, Stewart et al. '01)

It turns out that the effective Lagrangian we have constructed is *exact*, despite the fact that we did not consider loop corrections in it's construction.

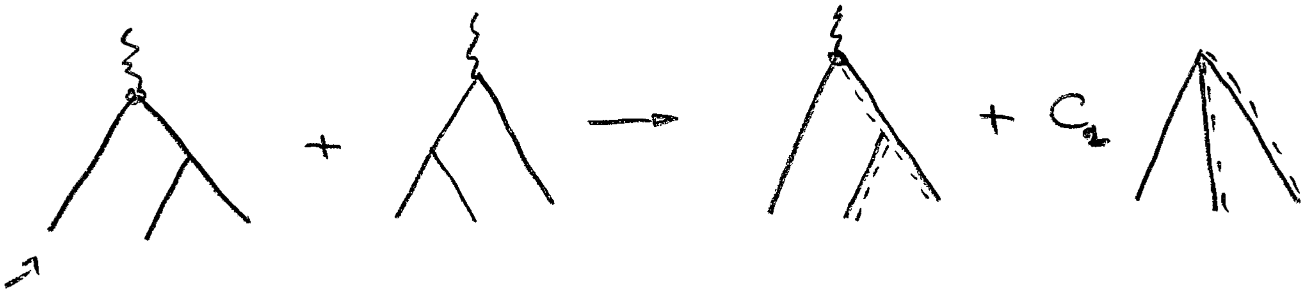
The reason is that each sector ($\mathcal{L}_s, \mathcal{L}_{c_1}, \mathcal{L}_{c_2}$) is equivalent to the full theory and all loop integrals with only c_1+s or c_2+s are scaleless and vanish.

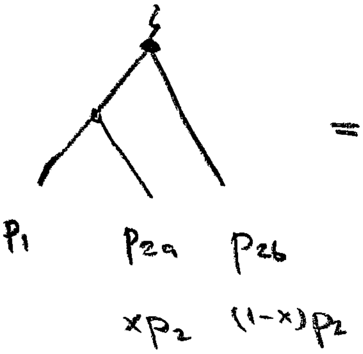
The only part with nontrivial matching is the current operator $A\phi^2$

At lowest order

$$\phi^2 \longrightarrow i\, \phi_{c_1}\phi_{c_2}$$

At O(g): $\phi^2 \longrightarrow \phi_{c_1}\phi_{c_2} + C_2\left(\phi_{c_1}\phi_{c_2}^2 + \phi_{c_1}^2\phi_{c_2}\right)$





$$= \frac{i}{(p_1+p_{2a})^2}\, ig = \frac{1}{2p_1^- \cdot p_{2a}^+}(-g)$$

$$\bar{n}\cdot p_1\, n\cdot p_{2a} \sim Q^2$$

$$\phi^2 \longrightarrow \phi_{c_1}\phi_{c_2} + g\left[\left(\frac{1}{\bar{n}\cdot\partial}\phi_{c_1}\right)\left(\frac{1}{n\partial}\phi_{c_2}\right)\phi_{c_2} + (1\leftrightarrow 2)\right]$$

$\nearrow$

$p^\mu \sim i\partial^\mu$

The appearance of an inverse derivative is at first sight disturbing. Note however that it corresponds to $\frac{1}{E}$ and by construction E is large.

An alternative way of writing the inverse derivative is as an integral

$$\frac{i}{in\partial + i\varepsilon}\phi(x) = \int_{-\infty}^{0} ds\, \phi(x+s\cdot n)$$

It is a characteristic feature of SCET that the operators are non-local along the directions of large light-cone momentum. To write down the most general SCET operator, one smears the fields along the light-cone

$$A\,\phi^2(x) \longrightarrow A\left[\mathcal{O}_2(x) + \mathcal{O}_3(x) + \dots\right]$$

$$\mathcal{O}_2 = \int ds \int dt \; C_2(s,t)\, \phi_{c_1}(x + s\bar{n})\, \phi_{c_2}(x + tn)$$

$$\mathcal{O}_3 = \int ds \int dt_1 \int dt_2 \; C_3(s,t_1,t_2)\, \phi_{c_1}(x+s\bar{n}) \times \phi_{c_2}(x+t_1\bar{n})\, \phi_{c_2}(x+t_2\bar{n}) + (1 \Leftrightarrow 2)$$

We found $C_2(s,t) = \delta(s)\,\delta(t) + \mathcal{O}(g^2)$

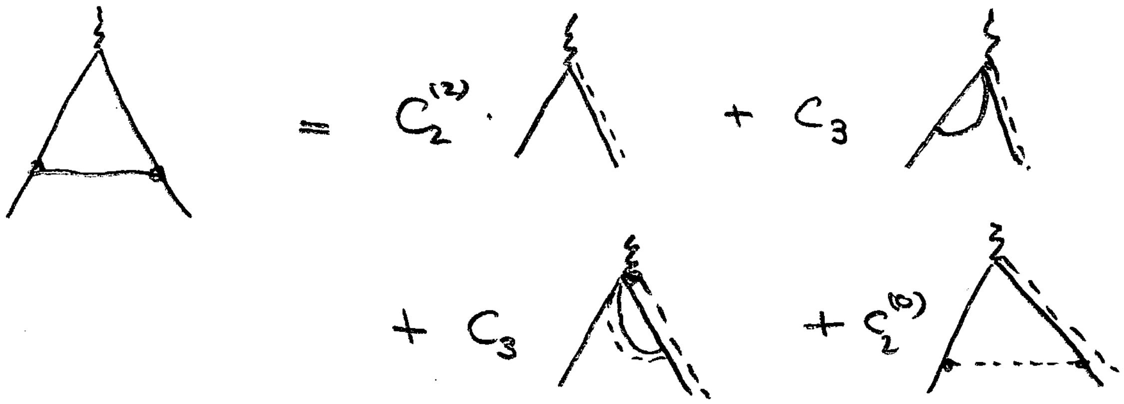
$$C_3(s,t_1,t_2) = g\,\Theta(-s)\,\Theta(-t_1)\,\delta(t_2) + \mathcal{O}(g^3)$$

The dependence on s, t is equivalent to dependence on the large energies in momentum space

$$\delta(s) \Leftrightarrow 1$$

$$\Theta(-s) \Longleftrightarrow \frac{1}{E_1}$$

Let us now check how the calculation of the ϕ^3 vertex diagram looks in the effective theorie



$C_2^{(0)}$ is the tree-level Wilson coefficient of O_2

$C_2^{(2)}$ it's one-loop value.

The four diagrams are in one-to-one contribution to the hard, coll-1, coll-2, and soft contributions we encountred when we did the strategy of region expansion earlier.

For order-by-order calculations, the strategy of regions is more efficient. SCET is useful to derive all-order properties such as factorization theorems. Furthermore the RG in the effective theory can be used to resum log's of $\frac{p^2}{E^2}$ to all orders.

7.2.1 Factorization of the Sudakov form factor

Let us use our scalar SCET for a factorization discussion. The four-dimensional theory has a dimensionful coupling, which complicates the analysis. Let's instead consider the theory in six dimensions.

$$S = \int d^dx \left[\frac{1}{2}(\partial_\mu \phi)^2 - g\,\phi^3\right]$$

$$\Downarrow$$

$$\longrightarrow [\phi] = \frac{d-2}{2} \qquad [\phi] = 1 \text{ in } d=4$$

$$[\phi] = 2 \text{ in } d=6$$

$$\longrightarrow [g] = 1 \text{ in } d=4$$

$$[g] = 0 \text{ in } d=6 \qquad \left(3\cdot\frac{d-2}{2} - d = \frac{d-6}{2}\right)$$

Now let's establish the power counting in the EFT, i.e. see how the various fields count:

$$\langle \phi_c(x)\,\phi_c(0)\rangle \sim \int d^6p \underbrace{e^{-ipx}}_{1} \frac{i}{p^2}$$

$$\sim \lambda^2\cdot\underbrace{\lambda^4}_{} \cdot \frac{1}{\lambda^2} = \lambda^4$$

$$\phi_c \sim \lambda^2$$

4 transverse directions.

$$\int d^6p = \frac{1}{2}\int dn\cdot p \int d\bar{n}\cdot p \int d^4p_\perp$$

$$\langle \phi_s(x)\,\phi_s(0)\rangle = \int d^6p\; e^{-ipx}\,\frac{i}{p^2} \sim (\lambda^2)^6 \cdot \frac{1}{\lambda^4} \sim \lambda^8$$

$$\phi_s \sim \lambda^4 .$$

Now let's look at terms in L

$$\int d^6x\; \frac{1}{2}(\partial_\mu\phi_c)^2 \sim \frac{1}{\lambda^2}\frac{1}{\lambda^4}\,\lambda^2\,(\lambda^2)^2 = \lambda^0 \quad \checkmark$$

$$\int d^6x\; \frac{1}{2}(\partial_\mu\phi_s) \sim \frac{1}{(\lambda^2)^6}\;\lambda^4\,(\lambda^4)^2 = \lambda^0$$

$$\int d^6x\; g\,\phi_c^3 \sim \frac{1}{\lambda^6}\,(\lambda^2)^3 = \lambda^0$$

$$\int d^6x\; g\,\phi_s^3 \sim \frac{1}{\lambda^{12}}\,(\lambda^4)^3 = \lambda^0$$

$$\int d^6x\; g\,\phi_c^2\,\phi_s \sim \underset{\uparrow}{\frac{1}{\lambda^6}}\,(\lambda^2)^2\,\lambda^4 = \underline{\underline{\lambda^2}} \quad \text{suppressed}$$

x scales collinear

Current operators

$$\underset{\frac{1}{\lambda^4}}{\int d^6x}\; A\;\underset{\lambda^2}{\phi_{c_1}}\,\underset{\lambda^2}{\phi_{c_2}} \sim 1$$

$$\underset{\frac{1}{\lambda^4}}{\int d^6x}\; A\,\underset{(\lambda^2)^2}{(\phi_{c_1})^2}\,\underset{\lambda^2}{\phi_{c_2}} \sim \lambda^2$$

$$\int d^6x\; A\,\phi_{c_1}\phi_{c_2}\phi_s \sim \lambda^4$$

In summary:

$$\int d^6x \, \mathcal{L}_{SCET} = \int d^6x \left[\mathcal{L}_{c_1} + \mathcal{L}_{c_2} + \mathcal{L}_s \right] + O(\lambda^2)$$

Current operator

$$\int d^6x \, A \, \phi^2_{(x)} \sim \int d^6x \int ds \int dt \; C(s,t) \, \phi_{c_1}(x+s\bar{n}) \, \phi_{c_2}(x+t\bar{n})$$

$$+ O(\lambda^2)$$

Since soft-collinear interactions are power suppressed, we now obtain a factorization theorem

$$G(p_1, p_2) = \int d^6x_1 \int d^6x_2 \; e^{-ip_1x_1 + ip_2x_2}$$

$$\langle 0 | T \{ \phi(x_1) \, A\phi^2(0) \, \phi(x_2) | 0 \rangle$$

$$= \int d^6x_1 \int d^6x_2 \; e^{-ip_1x_1 + ip_2x_2} \int ds \int dt \; C(s,t)$$

$$A \langle 0 | T \{ \phi_{c_1}(x_1) \, \phi_{c_1}(s\bar{n}) \} | 0 \rangle \langle 0 | T \{ \phi_{c_2}(tn) \phi(x_2) \} | 0 \rangle$$

Use translation invariance

$$\langle 0|T\{\phi_{c_1}(x)\,\phi_{c_2}(s\bar n)\}|0\rangle = \langle 0|T\{\phi_{c_1}(x-s\bar n)\,\phi_{c}(0)\}|0\rangle$$

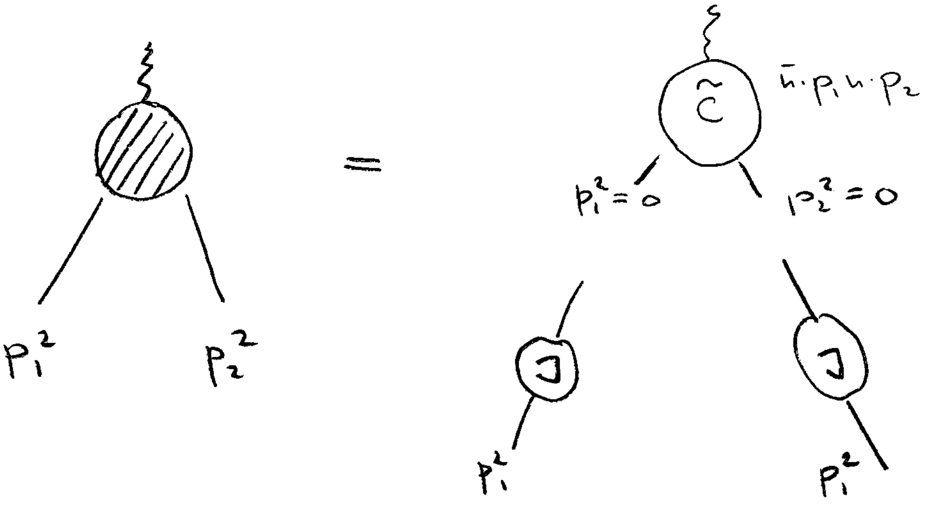
and translate $x_1 \to x_1 + s\bar n$, $x_2 \to x_2 - t\bar n$

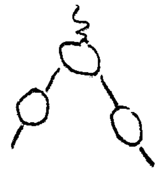
$$G(p_1, p_2) = \int ds \int dt \; C(s,t)\, e^{isp_1\cdot\bar n}\, e^{-itp_2 n}\; J(p_1^2)\, J(p_2^2)$$

$$\left[\; J(p_1^2) = \int d^6x \; e^{-ip_1 x} \langle 0|T\{\phi_{c_1}(x)\,\phi_{c_1}(0)\}|0\rangle\right.$$

$$\boxed{G(p_1 p_2) = \tilde C(\bar n\cdot p_1\; n\cdot p_2)\; J(p_1^2)\, J(p_2^2)}$$

We have factorized the Green's function into a product of a hard function $\tilde C$ and two jet-functions J. Note that $L_{c_1} \equiv L_{\phi^3}$, so we can calculate J using the full theory.



Note that one can always write  = , what makes our theorem nontrivial is that the form-factor part  is evaluated for $p_1^2 = p_2^2 = 0$. We have factorized the form-factor into a high-energy part $\tilde{C}(\bar{n}p_1, np_2)$ and a low energy contribution encoded in $J(p_1^2)$ and $J(p_2^2)$

It would be fun to use the theorem to resum Sudakov logarithms $\alpha_s^n \ln^{2n}\left(\frac{p_1^2 p_2^2}{Q^4}\right)$ to all orders using the renormalization group in the effective theory, but we will move on to QCD.

7.3. Generalization to QCD

The method for constructing the effective theory for QCD is analogous to the scalar case. In particular, the same momentum regions appear, since only the numerators of the diagrams differ between ϕ^3 and QCD.

The two complications with respect to the scalar case are that not all components of the quark and gluon fields have the same scaling and that we need to ensure gauge invariance, in particular also for the non-local operators.

To make things simpler, let's only consider one type of collinear field $p_c^\mu \sim (\lambda^2, 1, \lambda)$ for the moment. Let's look at the fermion field. Split

$$\psi_c(x) = \xi(x) + \eta(x) \quad ; \quad \xi = P_+ \psi_c = \frac{\not n \not{\bar n}}{4} \psi_c$$

$$\eta = P_- \psi_c = \frac{\not{\bar n} \not n}{4} \psi_c$$

<u>Note</u>: $\not n \xi = 0$; $\not{\bar n} \eta = 0$

$$P_+^2 = \frac{\not n \not{\bar n}}{4} \frac{\not n \not{\bar n}}{4} = - \frac{\not n \not{\bar n}}{4} \frac{\not{\bar n} \not n}{4} + \frac{\not n \not{\bar n}}{4} \frac{2 n \bar n}{4} = \frac{\not n \not{\bar n}}{4} = P_+$$

$$P_+ + P_- = \frac{\not n \not{\bar n}}{4} + \frac{\not{\bar n} \not n}{4} = \frac{2 n \cdot \bar n}{4} = 1 .$$

$$\langle 0 | T\{ \xi(x) \bar{\xi}(0) \} | 0 \rangle = \frac{\not{n}\not{\bar{n}}}{4} \langle 0 | T\{ \psi(x) \bar{\psi}(0) \} | 0 \rangle \frac{\not{\bar{n}}\not{n}}{4}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2+i\varepsilon} e^{-ipx} \underbrace{\frac{\not{n}\not{\bar{n}}}{4} \not{p} \frac{\not{\bar{n}}\not{n}}{4}}_{= \frac{\not{n}}{2} \bar{n}\cdot p} \sim \lambda^2 \lambda^2 \frac{1}{\lambda^2}$$

$$\Rightarrow \xi(x) \sim \lambda .$$

The same argument gives

$$\eta(x) \sim \lambda^2 .$$

For the soft quark field

$$q_s \bar{q}_s \sim (\lambda^2)^4 \frac{1}{\lambda^4} \lambda^2 = \lambda^6$$

$$q_s \sim \lambda^3$$

For the gluon field

$$\langle 0 | T\{ A^\mu(x) A^\nu(0) \} | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{i}{p^2} \left(-g^{\mu\nu} + \xi \frac{p^\mu p^\nu}{p^2} \right)$$

$$\rightsquigarrow A_c^\mu \sim p_c^\mu \quad ; \quad A_s^\mu \sim p_s^\mu$$

$$\rightarrow \bar{n}\cdot A_{c_1} \sim 1 \; ; \; n\cdot A_{c_1} \sim \lambda^2 \; ; \; A_\perp^\mu \sim \lambda$$

$$A_s^\mu \sim \lambda^2 .$$

The collinear fermion Lagrangian has a special form since the two components encoded in η are of a higher power than the ξ components and can be integrated out. With $\not{n}\xi = 0$, $\bar{\not{n}}\eta = 0$, we find

$$\mathcal{L}_c = \bar{\Psi}_c \, i\not{D}_c \, \Psi = (\bar{\xi} + \bar{\eta}) \left[\frac{\not{n}}{2} i\bar{n}\cdot D + \frac{\bar{\not{n}}}{2} in\cdot D + i\not{D}_\perp \right] (\xi + \eta)$$

$$= \bar{\xi} \frac{\bar{\not{n}}}{2} n\cdot D \, \xi + \bar{\xi} \, i\not{D}_\perp \eta + \bar{\eta} \, i\not{D}_\perp \xi + \bar{\eta} \frac{\not{n}}{2} \bar{n}\cdot D \eta$$

Since the action is quadratic, we can integrate out η exactly. A short-cut for obtaining the result is to plug the solution of the equation of motion back into $\mathcal{L}_c$. The EOM's are

$$\frac{\bar{\not{n}}}{2} n\cdot D \, \xi = - i\not{D}_\perp \eta$$

$$i\not{D}_\perp \xi = - \frac{\not{n}}{2} \bar{n}\cdot D \, \eta$$

$$\Rightarrow \quad \frac{\bar{\not{n}}}{2} i\not{D}_\perp \xi = - \frac{\bar{\not{n}}\not{n}}{2} \bar{n}\cdot D \eta = - \bar{n}\cdot D \eta$$

$$\Rightarrow \quad \eta = - \frac{\bar{\not{n}}}{2i\bar{n}\cdot D} i\not{D}_\perp \xi \; ; \; \bar{\eta} = - \bar{\xi} \, i\overleftarrow{\not{D}}_\perp \frac{\bar{\not{n}}}{2in\overleftarrow{D}}$$

Plug in:

$$\mathcal{L}_c = \bar\xi_c \frac{\not{n}}{2} n\cdot D\, \xi_c + \bar\xi\, i\not{D}_\perp \frac{1}{i\bar n\cdot D}\, i\not{D}_\perp \frac{\not{\bar n}}{2}\xi$$

$$+ \bar\xi\, i\overleftarrow{\not{D}} \frac{1}{i\bar n \overleftarrow{D}}\, i\not{D}_\perp \frac{\not{\bar n}}{2}\xi$$

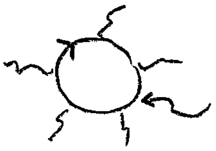
$$+ \bar\xi\, i\overleftarrow{\not{D}} \frac{\not{n}}{2 i\bar n\overleftarrow{D}} \underbrace{\frac{\not{\bar n}}{2}\, i\bar n D\, \frac{\not{n}}{2 i\bar n\overrightarrow{D}}}_{\frac{\not{n}\not{\bar n}}{4} \,\hat{=}\, 1} i\not{D}_\perp\, \xi$$

$$= \bar\xi_c \frac{\not{n}}{2} n\cdot D\, \xi_c + \bar\xi\, i\not{D}_\perp \frac{1}{i\bar n D + i\varepsilon}\, i\not{D}_\perp \frac{\not{\bar n}}{2}\xi$$

↖ sign will not matter.

$\bar n\cdot D \sim E$ is large

If one integrate out the field η, one further obtains a determinant $\det(i\bar n\cdot D)$. This corresponds simply to a trivial overall factor in the path integral. To see this, note that it is gauge invariant, but independent of the gluon field in light-cone gauge $\bar n\cdot A = 0$. ⌈Alternatively, it is easy to see that all loop diagrams



$\frac{1}{\bar n p + i\varepsilon}$

vanish, because all poles are on the same side of the $n\cdot p$ axis.⌋

While the collinear quark Lagrangian has this somewhat complicated form, the collinear gluon Lagrangian is just the QCD expression with $A_\mu \to A_\mu^c$.

The same is true for the soft Lagrangian

$$\mathcal{L}_s = \bar{q}_s i\not{D}_s q_s - \frac{1}{4}(F_{\mu\nu}^{sa})^2$$

where $iD_s = i\partial + A_s$

$$i g_s F_{\mu\nu}^a t^a = [iD_s^\mu, iD_s^\nu]$$

So our effective Lagrangian consists of several copies of QCD, for each collinear sector and for the ~~soft~~ momentum region.

What is still missing are the soft-collinear interactions. The general construction is somewhat complicated (see hep-ph/0211358 by Beneke and Feldmann) but we only need the leading power Lagrangian to derive factorization in the limit $E \to \infty$.

To get the leading interactions, remember that

$$(n\cdot A_{c_1}, \bar{n} A_{c_1}, A_{\perp}) \sim (\lambda^2, 1, \lambda)$$

$$(n A_s, n A_s, A_{s\perp}) \sim (\lambda^2, \lambda^2, \lambda^2)$$

$$\xi \sim \lambda \qquad q_s \sim \lambda^3$$

The leading soft-collinear interactions can be obtained by replacing $\phi_{c_1}(x) \to \phi_s(x_-)$ in the collinear Lagrangian

E.g. $$-\frac{g}{3!}\int d^4x\, \phi_{c_1}^3(x) \longrightarrow -\frac{g}{2!}\int d^4x\, \phi_{c_1}^2(x)\,\phi_s(x_-)$$

- Since q_s is of a higher power than ξ, interactions with soft quarks do not appear at leading order.

- Only the $n\cdot A_s$ component of the soft gluon field is not power suppressed, so only this component enters the leading soft-collinear interactions. So we can replace

$$A_c^\mu(x) \to (n\cdot A_c(x) + n\cdot A_s(x_-))\frac{\bar{n}^\mu}{2} + \bar{n}\cdot A_c(x)\frac{n^\mu}{2} + A_{c\perp}^\mu(x)$$

in the collinear Lagrangian.

To summarize, our Lagrangian is

$$\mathcal{L}_{SCET} = \bar{q}\, i\not{D}_s\, q + \bar{\xi}\frac{\not{\bar{n}}}{2}\left[n\cdot D + i\not{D}_{c\perp}\frac{1}{i\bar{n}\cdot D_c}i\not{D}_{c\perp}\right]\xi$$

$$-\frac{1}{4}\left(F^{sa}_{\mu\nu}\right)^2 - \frac{1}{4}\left(F^{ca}_{\mu\nu}\right)^2$$

where $iD_s = i\partial_\mu + g A_{s\mu}$

$iD_c = i\partial_\mu + g A_{c\mu}$

$in\cdot D = in\cdot\partial + g\, n\cdot A_c(x) + g\, n\cdot A_s(x_-)$

and $ig F^s_{\mu\nu} = [iD^s_\mu, iD^s_\nu]$

$ig F^c_{\mu\nu} = [iD_\mu, iD_\nu]\Big|_{D_\mu \to n\cdot D\frac{\bar{n}^\mu}{2} + \bar{n}\cdot D_c\frac{n^\mu}{2} + D^\mu_{c\perp}}$

I have only written the Lagrangian for one collinear sector, but in our applications we'll always have two sectors, $p_{c_1} \sim (\lambda^2, 1, \lambda)$ & $p_{c_2} \sim (1, \lambda^2, \lambda)$. The second collinear sector is obtained by replacing $n^\mu \leftrightarrow \bar{n}^\mu$ (which implies $x_+ \leftrightarrow x_-$) in the first sector.

Let us now discuss gauge invariance. In the same way we have expanded the Lagrangian, we will also expand the transformations. Furthermore, we will consider both soft and collinear gauge transformations.

Since the collinear transformations involve a field with large energy, the soft fields cannot transform under them: $V_c = \exp[i\alpha^a t^a]$

$$\xi_c \to V_c \xi_c \qquad q_s \to q_s$$

$$A_s^\mu \to A_s^\mu$$

$$A^\mu_{c\perp} \to V_c A^\mu_{c\perp} V_c^\dagger + \frac{i}{g} V_c \overset{/\!/\,(\partial^\mu_\perp V_c^\dagger)}{[\partial^\mu_\perp , V_c^\dagger]}$$

$$\bar{n} A_c \to V_c \bar{n} A_c V_c^\dagger + \frac{i}{g} V_c [\bar{n}\partial , V_c^\dagger]$$

$$n \cdot A_c \to V_c n A_c V_c^\dagger + \frac{i}{g} V_c [n \cdot D_s(\underset{\Uparrow\,!}{x_-}) , V_c^\dagger]$$

The last transformation law is special. It ensures that $in \cdot D = in\partial + gA_c(x) + gA_s(x_-)$ transforms as $V\, in \cdot D\, V^\dagger$.

Let's now consider soft transformations $V_s = \exp[i\alpha_s^a t^a]$.

The soft fields transform in the standard way

$$A_s^\mu \to V A_s^\mu V^+ + \frac{i}{g} V[\partial^\mu, V^+]$$

$$q_s \to V q_s$$

When transforming collinear fields, we must expand $\phi_c \phi_s \to \phi_c(x)\phi_s(x_-)$, therefore

$$\xi \to V_s(x_-)\xi$$

$$A_c^\mu \to V_s(x_-) A_c^\mu V_s^+(x_-)$$

Note that the "missing" $V_s[\partial^\mu, V^+]$ term is power suppressed for $A_{c\perp}^\mu$ and $\bar{n}\cdot A_c$. The small component of the collinear field, on the other hand, appears in the combination

$$n\cdot A_c(x) + n\cdot A_s(x_-)$$

$$\to V_s(x_-)\left[n\cdot A_c(x) + n\cdot A_s(x_-)\right]V_s(x_-)$$
$$+ \frac{i}{g} V_s(x_-)[n\cdot\partial, V_s(x_-)]$$

and transforms as expected.

It is easy to check that our Lagrangian is separately invariant under soft and collinear gauge transformations. The different covariant derivatives all transform as $V iD_\mu V^\dagger$ and the fermions as Vq and $V\xi$, with $x \to x_-$ in the appropriate places.

7.3.1. Wilson lines and decoupling transformation

When matching the current operator in ϕ^3, we encountered non-local operators of the form $\phi_{c_1}(x+s\bar{n})\,\phi_{c_2}(x+tn)$. In a gauge theory a product of fields at different points is only gauge invariant if they are connected by a Wilson line. Define

the color matrices are ordered along the path.

$$[x+s\bar{n},\, x] = \mathbb{P}\exp\left[ig\int_0^s ds'\, \bar{n}\cdot A(x+s'\bar{n})\right]$$

This object transforms as*

$$[x+s\bar{n},\, x] \rightarrow V(x+s\bar{n})\,[x+s\bar{n},\, x]\, V^{+}(x)$$

So, an operator such as

$$\psi(x+s\bar{n})\,[x+s\bar{n},\, x]\;\psi(x)$$

is gauge invariant.

---

* This is the transformation law of the link field $U(x,y)$ we used when we discussed gauge transformations.

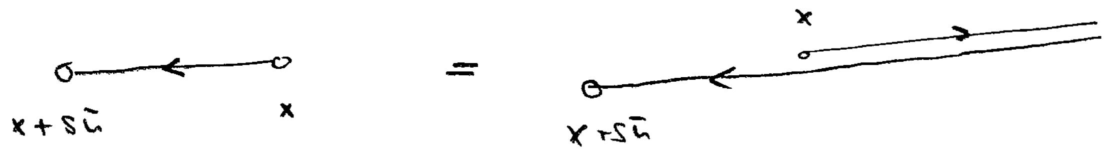
Instead of a line from x to $x + s\bar{n}$, it is customary to work with Wilson lines which run to infinity

$$W(x) = \mathbb{P} \exp\left[ig \int_{-\infty}^{0} ds \; \bar{n}\cdot A(x + s\bar{n}) \right]$$

$$= [x, -\infty]$$

A finite segment is

$$[x + s\bar{n}, x] = W(x + s\bar{n})\, W^{+}(x)$$



If one restricts oneself to gauge transformations which vanish at infinity, the products

$$\chi(x) = W^{+}(x)\,\psi(x) \quad \text{and} \quad \bar{\chi}(x) = \bar{\psi}(x)\, W(x)$$

are gauge invariant and can be used as building blocks to construct non-local operators.

The Wilson line $W(x)$ fulfills

$$i\bar{n}\cdot D\, W(x) = 0 .$$

$$i\bar{n}\cdot D\, W(x) = (i\bar{n}\partial + g\bar{n}\cdot A)\,\mathbb{P}\exp\left[ig\int_{-\infty}^{0} ds\, \bar{n}A(x+s\bar{n})\right]$$

$$= (i\bar{n}\partial + g\bar{n}A)\,\mathbb{P}\exp\left[ig\int_{-\infty}^{n\cdot x/2} ds\, \bar{n}A\left(s\bar{n}^\mu + \bar{n}\cdot x\frac{n^\mu}{2} + x_\perp^\mu\right)\right]$$

$$= \left[i\frac{\bar{n}\cdot n}{2}\, ig\bar{n}A(x) + g\bar{n}A(x)\right] W(x) = 0$$

Two types of Wilson lines are important in SCET:

Collinear $\quad W_c(x) = \mathbb{P}\exp\left[ig\int_{-\infty}^{0} ds\, \bar{n}A_c(x+\bar{n}s)\right]$

Soft $\quad S_n(x) = \mathbb{P}\exp\left[ig\int_{-\infty}^{0} ds\, n\cdot A_s(x+ns)\right]$

The collinear lines are used to build operators, while the soft lines are useful because of the structure of the soft interaction.

The interaction between quarks and soft gluons has the form

$$\mathcal{L}_{c+s} = \bar{\xi}\,\frac{\not{\bar{n}}}{2}\, in\cdot D\, \xi$$

$$in\cdot D = in\cdot\partial + g n A_c(x) + g n\cdot A_s(x_-) \quad ; \; x_- = \bar{n}\cdot x\,\frac{n^\mu}{2}$$

Now perform a field redefinition:

$$\xi_c(x) \longrightarrow S_n(x_-)\,\xi_c^{(0)}(x)$$
$$A_c^\mu(x) \longrightarrow S_n(x_-)\,A_c^{(0)\mu}(x)\,S^\dagger(x_-)$$

$$\mathcal{L}_{c+s} = \bar{\xi}^{(0)} S_n^\dagger(x_-)\,\frac{\not{\bar{n}}}{2}\, n_\mu\cdot\left(i\partial^\mu + A_s^\mu(x_-) + S_n(x_-)A_c^\mu(x)S_n^\dagger(x_-)\right)S\xi$$

$$= \bar{\xi}^{(0)}\, S_n^\dagger(x_-)\,S_n^\dagger(x_-)\,\frac{\not{\bar{n}}}{2}\left[in\partial + g A_c^{(0)}(x)\right]\xi^{(0)}$$

$$= \bar{\xi}^{(0)}\; in\cdot D_c^{(0)}\,\frac{\not{\bar{n}}}{2}\,\xi^{(0)}$$

This is also called the decoupling transformation, since it decouples the soft gluons from the collinear Lagrangian.

This decoupling is an important ingredient to factorization, but does not imply that everything factorizes at leading power.

For example, the vector current in QCD matches at leading power onto

$$J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$$

$$\longrightarrow \int ds \int dt \; C(s,t)\,\bar{\chi}_1(x+s\bar{n})\,\gamma_\perp^\mu\,\chi_2(x+tn)$$

$$\Bigg[\; \chi_1 = W_{c_1}^\dagger \cdot \xi_{c_1} \qquad \not{n}\chi_1 = 0$$

$$\chi_2 = W_{c_2}^\dagger\, \xi_{c_2} \qquad \not{\bar{n}}\chi_2 = 0$$

$$\gamma^\mu = \not{n}\frac{\bar{n}^\mu}{2} + \not{\bar{n}}\frac{n^\mu}{2} + \gamma_\perp^\mu \;\Bigg]$$

The decoupling transformations are

$$\chi_1(x) \longrightarrow S_n(x_-)\,\chi_1^{(0)}(x)$$

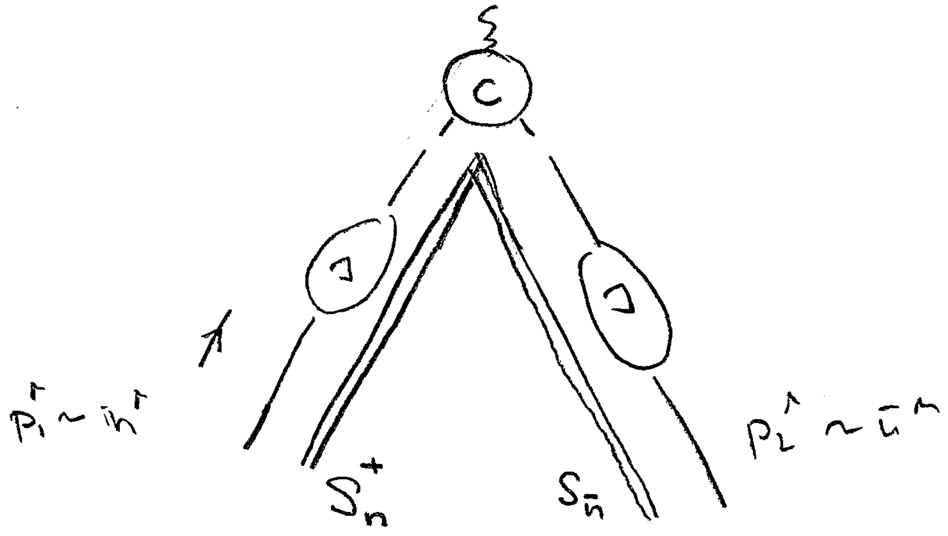
$$\chi_2(x) \longrightarrow S_{\bar{n}}(x_+)\,\chi_2^{(0)}(x)$$

(for χ_2, we have to replace $n \leftrightarrow \bar{n}$)

So the current becomes

$$J^\mu(x) = \int ds \int dt\, C(s,t)\, \bar{\chi}_1^{(0)}(x+s\bar{n})\, S_n^+(x_-)\, S_{\bar{n}}(x_+)\, \gamma_\perp^\mu\, \chi_2^{(0)}(x)$$

We managed to decouple the soft fields from the Lagrangian but they are still present in the operator J^μ. In contrast to ϕ^3 in $d=6$, the Sudakov form factor gets low energy contributions which describe a long-range interaction between the fast-moving in- and out-going quarks.



In this sence, it is non-factorizable: the low-energy contributions do not separate into corrections associated with the individual particles.

8.1. Factorization of the DIS cross section

Let us now analyze the process in SCET and derive a factorization theorem for the cross section. It is most convenient to work in the Breit frame, where

$$q^\mu = \frac{Q}{2}\left(\bar{n}^\mu - n^\mu\right) = Q\,(0,0,0,-1)$$

$$q^2 = -Q^2 \quad \checkmark$$

$$P^\mu = \bar{n}\cdot P\,\frac{n^\mu}{2} + \frac{M_P^2}{\bar{n}\cdot P}\,\frac{\bar{n}^\mu}{2}$$

$$P^2 = M_P^2\,\frac{2\bar{n}\cdot n}{4} = M_P^2 \quad \checkmark$$

$$x = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{\bar{n}\cdot P} + O(m_P^2/Q^2)$$

Recall that

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right)W_1 + \left(P^\mu - \frac{q^\mu q\cdot P}{q^2}\right)\left(P^\nu - \frac{q^\nu q\cdot P}{q^2}\right)W_2$$

$$= \left[-g^{\mu\nu} + \frac{(\bar{n}^\mu - n^\mu)(\bar{n}^\nu - n^\nu)}{4}\right]W_1 + \frac{(\bar{n}\cdot P)^2}{16}\,(n^\mu + \bar{n}^\mu)(n^\nu + \bar{n}^\nu)\,W_2$$

The tensor $W^{\mu\nu}$ is defined as

$$W^{\mu\nu} = \frac{1}{2}\sum_s \langle P,s \,|\, T^{\mu\nu} \,|\, P,s \rangle$$

$$T^{\mu\nu} = i\int d^4x \; e^{iqx} \, T[\, J^+_\mu(x) \, J_\nu(0) \,]$$

One way to analyze $T^{\mu\nu}$ would be to expand it in a series of local operators, however because we later take the proton matrix element operators such as

$$\frac{1}{Q^n}\langle \bar{q}(\bar{n}\cdot D)^n q\rangle \sim \left(\frac{\bar{n}\cdot p}{Q}\right)^n = x^n$$

are not suppressed. This situation, where we have very energetic particles, is precisely what SCET was designed for. Because the $\bar{n}\cdot\partial$ derivatives on collinear fields are not suppressed the operators are non-local along the corresponding light-cone direction. To analyze $T^{\mu\nu}$ in this effective theory, we now write down the most general

Leading power operator in the EFT.

First, since we average over spins, we can write

$$T^{\mu\nu} = \left(-g^{\mu\nu} + \frac{(\bar{n}^\mu - n^\mu)(\bar{n}^\nu - n^\nu)}{4}\right) T_1$$

$$\frac{(\bar{n}\cdot P)^2}{16} \left(n^\mu + \bar{n}^\mu\right)\left(n^\nu + \bar{n}^\nu\right) T_2$$

where T_1 and T_2 are scaler operators in SCET.

⌜ It is a bit strange to include a factor $\bar{n}\cdot P$ in the operator $T^{\mu\nu}$, since it does not know about the proton. We do it so that

$$\langle p | T_1 | p \rangle = W_1 \quad ; \quad \langle p | T_2 | p \rangle = W_2 .$$ ⌟

So now we should write down operators. As building blocks, we use

$$\chi(x) = W_c^\dagger \xi_c(x)$$

$$\mathcal{B}_\perp^\mu(x) = W_c^\dagger(x)\, \bar{n}_\nu \left[G_c^{\nu\mu}(x) - G_c^{\nu\alpha}(x)\, n^\alpha \frac{\bar{n}^\mu}{2} \right] W_c(x)$$

Remember that $A_c^\mu \sim (n\cdot A, \bar{n} A, A_\perp^\mu)$

$\sim (\lambda^2, 1, \lambda)$

Thus $\bar{n}_\nu G^{\nu\mu}_\perp \sim \lambda$

$\bar{n}_\nu n_\mu G^{\mu\nu} \sim \lambda^2$

$G_\perp^{\mu\nu} \sim \lambda^2$

So at leading power only $\mathcal{B}_\perp^\mu$ can enter.

The lowest order operator is $\mathbb{1}$, however it does not have an imaginary part. There are no scalar operators with one field. The first nontrivial possibilities are

$$O_1 = \bar{\chi}(s\bar{n}) \frac{\not{\bar{n}}}{2} \chi(t\bar{n}) \sim O(\lambda^2)$$

$$O_2 = \mathrm{tr}\left[\mathcal{B}_\perp^\mu(s\bar{n})\, \mathcal{B}_\perp^\mu(t\bar{n})\right] \sim O(\lambda^2)$$

Note that $\not{n}\chi = 0$, $\frac{\not{n}\not{\bar{n}}}{4}\chi = \chi$:

$$\bar{\chi}\chi = \bar{\chi}\frac{\not{n}\not{\bar{n}}}{4}\chi = 0$$

$$\bar{\chi}\not{n}\chi = 0$$

$\bar{\chi}\gamma_5\frac{\not{\bar{n}}}{2}\chi$ violates parity.

So $\mathcal{O}_1$ and $\mathcal{O}_2$ are the only operators at leading power. ~~Because~~ of translation invariance only the distance between the fields matters, so we can set $t=0$ without loss of generality. So, up to power suppressed terms

$$T_1 = \int ds \Big\{ C_{1q}(s, Q^2)\, \bar{\chi}(s\bar{n}^\mu) \frac{\bar{n}\!\!\!/}{2} \chi(0) + C_{1g}(s, Q^2)\, \mathrm{tr}\big[\mathcal{B}_\perp^\alpha (s\bar{n}^\mu) \mathcal{B}_{\perp\alpha}(0) \big] \Big\}$$

and analogously for T_2, with different coefficients $C(s, Q^2)$

Now we decouple soft gluons $\quad x^- = \bar{n}\cdot x \frac{n^\mu}{2}$

$$\chi(x) \longrightarrow S_n(x_-)\, \chi^{(0)}(x)$$

it follows $\quad \chi(s\bar{n}) \longrightarrow S_n(0)\, \chi^{(0)}(x)$

$$\mathcal{B}_\perp^\mu(x) \longrightarrow S(x_-)\, \mathcal{B}_\perp^\mu(x)\, S^\dagger(x_-)$$

So $\bar{\chi}(s\bar{n})\frac{\bar{n}\!\!\!/}{2}\chi(0)$

$$\longrightarrow \bar{\chi}^{(0)}(s\bar{n})\frac{\bar{n}\!\!\!/}{2}\underbrace{S_n^{\dagger}(0)S_n(0)}_{=1}\chi(0)$$

Similarly, the soft gluons decouple from tr$[B_\perp^\alpha B_\alpha]$.

So we have factorized the cross section!

Now we just need to understand what the factorization theorem means .... To get an understanding, let's calculate $\hat{W}_{q1,2}$, the <u>quark</u> matrix element of $T_{1,2}$:

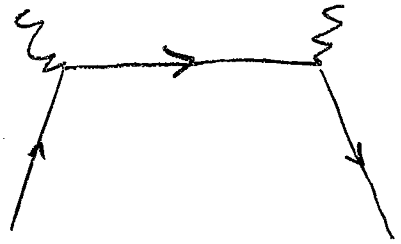
$$\frac{1}{2}\sum_{\text{spin}}\langle q(p)|T_1|q(p)\rangle =$$

$$\int ds\, C_{1q}(s,Q^2)\, e^{is\bar{n}\cdot p}\,\frac{1}{2}\sum_{\text{spin}}\bar{u}(p)\frac{\bar{n}\!\!\!/}{2}u(p)$$

$$= \tilde{C}_{1q}(\bar{n}\cdot p,Q^2)\,\frac{1}{2}\,\mathrm{tr}\Big[\overset{\approx \bar{n}\cdot p\frac{n\!\!\!/}{2}}{\overset{\shortparallel}{p\!\!\!/}}\,\frac{\bar{n}\!\!\!/}{2}\Big] = \tilde{C}_{1q}(\bar{n}\cdot p,Q^2)\,\bar{n}\cdot p$$

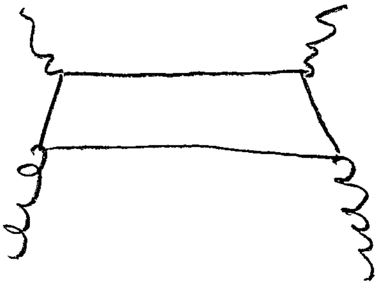
So $\tilde{C}_{1q}(\bar{n}\cdot p, Q^2)\,\bar{n}\cdot p = \hat{W}_{1q}(\bar{n}\cdot p, Q^2)$.

The Fourier transformed Wilson coefficient is just the partonic process



$\Rightarrow \tilde{C}_{1q}\,,\ \tilde{C}_{2g}$

The gluonic coefficient is obtained from



+ others $\Rightarrow C_{1g}\,, C_{2g}$ are $O(\alpha_s)$

Next, let's consider the proton matrix elements.

One defines

$$\frac{1}{2}\sum_{\text{spin}} \langle P \,|\, \bar{\chi}(s\bar{n})\frac{\not{\bar{n}}}{2}\chi(0)\,|\, P\rangle$$

$$= \bar{n}\cdot P \int d\xi\, f_q(\xi)\, e^{i\xi\bar{n}\cdot Ps}$$

and

$$\frac{1}{2}\sum_{\text{spin}} \langle P |\, B_\perp^{\alpha}(s\bar{n})\, B_{\perp\alpha}(0)\, | P \rangle$$

$$= \int d\xi\ \xi(\bar{n}\cdot P)^2 f_g(\xi)\, e^{i\xi \bar{n}\cdot P s}$$

The integrations run from $-1 < \xi < 1$. The function $f_q(\xi)$ picks up a contribution when the field χ carries away a fraction ξ of the proton's momentum. Negative values correspond to the anti-particle case

$$\bar{f}_q(\xi) = f_{\bar{q}}(\xi) = -f_q(-\xi)$$

$$f_g(\xi) = f_g(-\xi) = \bar{f}_g(\xi)$$

Now plug the expressions for the matrix elements into the expressions for W_1 and W_2:

$$W_1 = \int ds\, C_{1q}(s, Q^2) \int d\xi\, \bar{n}\cdot p\, e^{i\xi\bar{n}\cdot p} f_q(\xi) + \text{"glue"}$$

$$= \int_{-1}^{1} d\xi\, \tilde{C}_{1q}(\xi\bar{n}\cdot p, Q^2)\, \bar{n}\cdot p\, f_q(\xi) + \text{"glue"}$$

$$= \int_{-1}^{1} \frac{d\xi}{\xi}\, \hat{W}_{1q}(\xi\bar{n}\cdot p, Q^2)\, f_q(\xi) + \text{"glue"}$$

$$\boxed{\begin{aligned} W_i &= \sum_q \int_0^1 \frac{d\xi}{\xi}\, \hat{W}_{iq}(\xi\bar{n}\cdot p, Q^2)\left[f_q(\xi) + f_{\bar{q}}(\xi)\right] \\ &\quad + \int_0^1 \frac{d\xi}{\xi}\, \hat{W}_{ig}(\xi\bar{n}\cdot p, Q^2)\, f_g(\xi) \quad ; \; i = 1, 2 \end{aligned}}$$

$$\frac{d^2\sigma}{dx\,dy} = \frac{2\alpha^2 y}{(Q^2)^2}\left[Q^2\, \mathrm{Im}\, W_1 + \frac{s^2}{2}(1-y)\, \mathrm{Im}\, W_2\right]$$

The hadronic part is real, so that the imaginery part of W_1 and W_2 is obtained by taking the imaginary part of the partonic $\hat{W}_{iq}$ and $\hat{W}_{ig}$.

We are summing over different quark flavors $q = u, d, s, \ldots$

To summarize: we have factored the cross section into a set of non-perturbative parton distribution functions (PDFs) $f_i(\xi)$ which are convoluted with partonic cross section. To lowest order in α_s, we reproduce the parton model result, but our formula gives operator definitions for the PDFs and allows us to systematically include higher orders in α_s.

8.2. Parton Distribution Functions (PDFs)

To calculate DIS or any hadron-collider cross-section we need the non-perturbative PDFs as input.

We now derive some properties of these functions and then discuss their determination

The PDFs fulfill the following sum rules

Momentum sum rule:

$$\sum_i \int_0^1 d\xi \, \xi \, f_i(\xi) = 1$$

Flavor conservation. For a proton

$$\int_0^1 d\xi \sum_i \left(f_u(\xi) - f_{\bar{u}}(\xi) \right) = 2$$

$$\int_0^1 d\xi \sum_i \left(f_d(\xi) - f_{\bar{d}}(\xi) \right) = 1$$

$$\int_0^1 d\xi \sum_i \left(f_s(\xi) - f_{\bar{s}}(\xi) \right) = 0$$

In the parton-model interpretation, these are simple to understand. The momentum sum rule simply states, that the total momentum of the proton is given by the sum of all parton momenta.

The flavor sum rules state that the proton contains two u-quarks and $\bar{u}u$ pairs, etc.

Let us derive the flavor sum rule from the operator definition:

$$f_q(\xi) = \int \frac{ds}{2\pi} e^{-i\bar{n}\cdot P\xi s} \langle p | \bar{\chi}_q(s\bar{n}) \frac{\not{n}}{2} \chi(0) | p \rangle$$

$$= \int \frac{ds}{2\pi} e^{-i\bar{n}\cdot P\xi s} \langle p | \bar{q}(s\bar{n}) \frac{\not{n}}{2} [s\bar{n}, 0] \, q(0) | p \rangle$$

We have

$$f_{\bar{q}}(\xi) = - f_q(-\xi)$$

So

$$\int_0^1 d\xi\,\left[f_q(\xi) - f_{\bar q}(\xi)\right]$$

$f_q(\xi) = 0$ for $|\xi|>1$

$$= \int_{-1}^{1} d\xi\, f_q(\xi) = \int_{-\infty}^{\infty} d\xi\, f_q(\xi)$$

$$= \int d\xi \int \frac{ds}{2\pi}\, e^{i\bar n P \xi s} \langle p|\, \bar q(s\bar n) \frac{\bar{n}\!\!\!/}{2} [s\bar n, 0]\, q(0)\, |p\rangle$$

$$= \frac{1}{\bar n \cdot P} \langle p|\, \bar q(0) \frac{\bar{n}\!\!\!/}{2} q(0)\, |p\rangle$$

$$\langle p|\, \bar q \gamma^0 q\, |p\rangle = (\#q\text{'s} - \#\bar q\text{'s}) \cdot 2E$$

$$\langle p|\, \bar q \gamma^\mu q\, |p\rangle = (\#q\text{'s} - \#\bar q\text{'s})\, 2P^\mu$$

$$\Rightarrow \int_0^1 d\xi\, \left[f_u(\xi) - f_{\bar u}(\xi)\right] = 2$$

So we have derived the flavor sum rules. Note that an integral over the PDF lead to a local operator.

More generally, let us consider the moments

$$M_q^N = \int_0^1 \frac{d\xi}{\xi}\, \xi^N \left[f_q(\xi) + (-1)^N f_{\bar q}(\xi) \right]$$

$$= \int_{-\infty}^{\infty} \frac{d\xi}{\xi}\, \xi^N f_q(x)$$

$$= \int_{-\infty}^{\infty} d\xi\, \xi^{N-1} \int \frac{ds}{2\pi}\, e^{-i\bar n\cdot p \xi s} \langle p|\, \bar\chi(s\bar n) \frac{\bar{\not n}}{2} \chi(0) |p\rangle$$

$$= \int_{-\infty}^{\infty} d\xi \int \frac{ds}{(2\pi)} \left[\left(\frac{i\partial_s}{\bar n\cdot p} \right)^{N-1} e^{-i\bar n\cdot p \xi s} \right] \langle p|\, \bar\chi(s\bar n) \frac{\bar{\not n}}{2} \chi(0) |p\rangle$$

$$= \int_{-\infty}^{\infty} d\xi \int \frac{ds}{2\pi}\, e^{i\bar n\cdot p \xi s} \left(\frac{-i\partial_s}{\bar n\cdot p} \right)^{N-1} \langle p|\, \bar\chi(s\bar n) \frac{\bar{\not n}}{2} \chi(0) |p\rangle$$

$$(-i\partial_s)\, \bar q(s\bar n) [s\bar n, 0] q(0) \Big|_{s=0}$$

$$= \{ -i\partial_s \bar q(s\bar n) \} [s\bar n, 0] q(0) + \bar q(s\bar n)\, g A(s\bar n) [s\bar n, 0] q(0)$$

$$= \bar q(s\bar n) (-i\bar n\cdot \overleftarrow{D}) [s\bar n, 0] q(0)$$

$$= \left(\frac{1}{\bar n\cdot p} \right)^{N-1} \langle p|\, \bar q(0) (i\bar n\cdot D)^{N-1} \frac{\bar{\not n}}{2} q(0) |p\rangle$$

So moments of the PDFs correspond to local operators which can be calculated using lattice gauge theory. Unfortunately only $N = 2, 3$ is feasible. The reason is that these operators need to be renormalized. On the lattice with spacing a

$$\langle p | \bar{q} (iD)^{N-1} \frac{\not{n}}{2} q | p \rangle \sim \left(\frac{1}{a} \right)^{N-2}$$

the strong divergence makes it hard to extract the finite piece numerically.

Not only the local operators, also the non-local PDF operators need to be renormalized.

For the PDFs

$$\begin{pmatrix} f_q^{ren}(\xi, \mu) \\ f_g^{ren}(\xi, \mu) \end{pmatrix} = \int_\xi^1 \frac{dz}{z} \begin{pmatrix} Z_{qq}(z) & Z_{qg}(z) \\ Z_{gq}(z) & Z_{gg}(z) \end{pmatrix} \begin{pmatrix} f_q(\frac{\xi}{z}) \\ f_g(\frac{\xi}{z}) \end{pmatrix}$$

This is the usual $O^{\text{ren}} = Z \; O^{\text{bare}}$ relation.

Since we have two operators, they <u>mix</u> under renormalization and Z is a matrix. Furthermore, also the operators of different ξ mix, which gives the convolution.

The renormalized PDFs depend on the renormalization scale. From the fact, that the bare PDFs are scale independent, one obtains the renormalization group equation

$$\frac{d}{d\ln\mu}\begin{pmatrix} f_q^{\text{ren}}(\xi,\mu) \\ f_g^{\text{ren}}(\xi,\mu) \end{pmatrix} = \int_0^1 \frac{dz}{z} \overbrace{\begin{pmatrix} P_{qq}(z) & P_{qg}(z) \\ P_{gq}(z) & P_{gg}(z) \end{pmatrix}}^{\Gamma} \begin{pmatrix} f_q^{\text{ren}}(\frac{\xi}{z},\mu) \\ f_g(\frac{\xi}{z},\mu) \end{pmatrix}$$

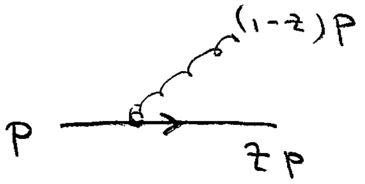
$\Gamma = - Z \otimes \frac{d}{d\ln\mu} Z^{-1}$ is the anomalous dimension which governs the scale dependence.

While we cannot calculate the PDFs in perturbation theory, we can calculate the scale dependence.

The above equation is called the DGLAP equation. (Altarelli, Parisi; Gribov, Lipatov; Dokshitzer '77)

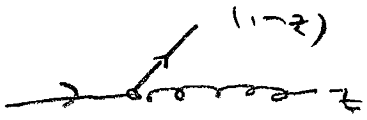
The kernels are $P_{ij}(z) = \frac{\alpha}{2\pi} P_{ij}^{(0)}(z) + \ldots\ldots$

$$P_{qq}^{(0)} = C_F \left[\left(\frac{1+z^2}{1-z} \right)_+ + \frac{3}{2} \delta(1-z) \right]$$

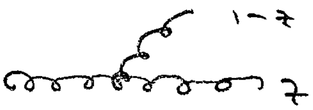


$$P_{qg}^{(0)} = T_F \left(z^2 + (1-z) \right)$$

$$P_{gq}^{(0)} = C_F \; \frac{1+(1-z)^2}{z}$$



$$P_{gg}^{(0)} = C_A \left[z \left(\frac{1}{1-z} \right)_+ + \frac{1-z}{z} + z(1-z) + \beta_0 \delta(1-z) \right]$$



They are also called splitting functions and are known to NNLO (Moch, Vermaseren and Vogt '04)

To solve the evolution equation, people usually go to moment space

$$M_i^N(\mu) = \int d\xi \, \xi^{N-1} f_i(\xi,\mu)$$

Then the problem reduces to the solution of the RG equation for the local operators.

At the end, one has to transform back:

$$f_i(\xi,\mu) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dN \, x^{-N} \, M_i^N(\xi,\mu)$$

Numerically, this is tricky, but there exists commercial grade numerical code to do it.

Let us now discuss the determination of the PDFs. We had obtained

$$\frac{d^2\sigma}{dx\,dy} = \frac{2\pi\alpha^2 s}{Q^4}\left[1+(1-y)^2\right]\sum_q e_q^2\, x\, f_q(x,\mu)$$

Define $F_2(x) = \sum_q e_q^2 f_q(x,\mu)$.

The natural choice for μ is $\mu = Q$.

Measuring $F_2(x)$ determines one linear combination of PDFs.

$$F_2 = x\left[\frac{4}{9} f_u(x) + \frac{1}{9} f_d(x)\right] + (\text{"sea-quarks"})$$

To get more information, scatter on neutron and use isospin

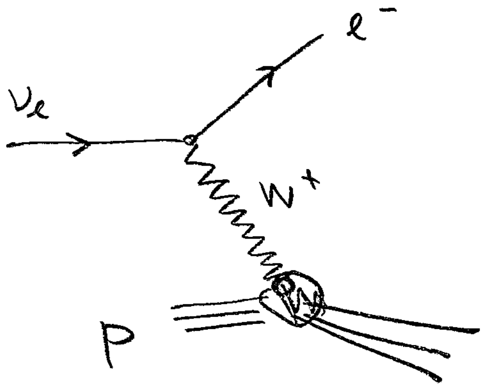
$$F_2^n = x\left(\frac{4}{9} f_u^n(x) + \frac{1}{9} f_d^n(x)\right] + (\text{"sea"})$$

$$= x\left[\frac{4}{9} f_d(x) + \frac{1}{9} f_u(x)\right] + (\text{"sea"})$$

(Experimentally one uses deutrons

$$F_2^d \cong \frac{1}{2}\left(F_2^p(x) + F_2^n(x)\right)$$

We also need the PDFs of s-quarks, and u-, d-, s-anti-quarks. To get these, we need something which interacts differently with quarks and anti-quarks. Use W's:



Finally, we need the gluon distribution. It can be inferred by measuring at different Q^2 and relating the PDFs by evolution.

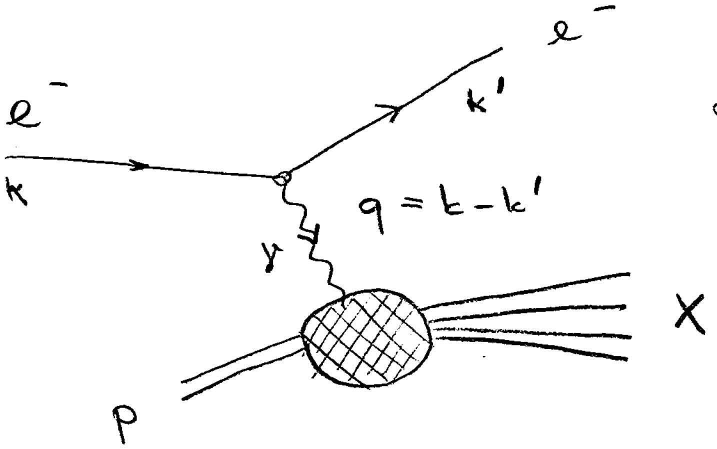
In practice, people start with a parametrization of the PDFs at some scale Q_0, e.g.

$$f_q(\xi, Q_0) = A_q x^{a_q}(1-x)^{b_q}\left[1 + c_q\sqrt{x} + \dots\right]$$

Then they evolve it to all Q^2 where exp. data is available and do a global fit to determine all parameters. There are a number of groups (MSTW, CTEQ, NNPDF, Alekhin, ...) doing such fits, and modern results also include a way to estimate the uncertainty. See figures for some results.

8. Deep inelastic scattering (DIS)

DIS refers to the process $e^- p \to e^- + X$ at high energies. At low energy, the scattering is elastic $e^- p \to e^- p$, but at high energies, one ends up with with a lot of hadrons.



$$Q^2 = -q^2$$

The amplitude is $\left(J^\mu_{e.m} = \sum_q e_q \bar{q} \gamma^\mu q \right)$

$$i\mathcal{M}(ep \to eX) = (-ie)\,\bar{u}(k')\gamma^\mu u(k)\,\frac{-i}{q^2}\, ie \int d^4x \sum_X e^{iq\cdot x} \langle X |\, J^\mu_{e.m}(x)\, | p \rangle$$

As in the case of $e^+e^- \to X$, let's use the optical theorem to rewrite the cross section.

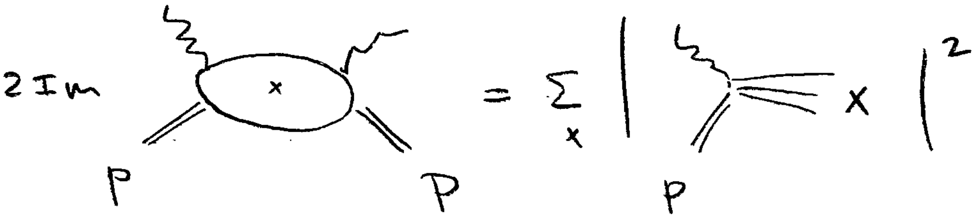
Consider

$$W^{\mu\nu}(q,P) = i\int d^4x\, e^{iqx} \langle P|T\{J^\mu_{em}(x), J^\nu_{em}(x)\}|P\rangle$$

averaged over proton spins. From this matrix element, we obtain the Compton scattering amplitude:

$$i\mathcal{M}(\gamma p \to \gamma p) = (ie)^2\, \varepsilon^*_\mu\, \varepsilon_\nu\, (-iW^{\mu\nu}(q,P))$$

$$2\,\mathrm{Im}\,\mathcal{M}(\gamma p \to \gamma p) = \sum_X |\mathcal{M}(\gamma p \to X)|^2$$



or

$$2\,\mathrm{Im}\, W^{\mu\nu} = \sum_X \langle P|\tilde{J}^\mu(-q)|X\rangle\langle X|\tilde{J}^\nu(q)|P\rangle$$

Now consider the DIS cross section

$$\sigma = \frac{1}{2s} \int \frac{d^3k'}{2E_k (2\pi)^3} \sum_X | \mathcal{M}(e(k) p(P) \to e^-(k') X(p_X)) |^2$$

Neglect m_e, m_p

$$= \frac{1}{2s} \int \frac{d^3k'}{2E_k (2\pi)^3} \; e^4 \frac{1}{(Q^2)^2} \; \frac{1}{2} \sum_{\text{spins}} \bar{u}(k) \gamma_\mu u(k') \bar{u}(k') \gamma_\nu u(k)$$

$$\cdot \; 2 \, \mathrm{Im} \, W^{\mu\nu}(P,q) .$$

$$\frac{1}{2} \sum_{\text{spins}} \bar{u}(k) \gamma_\mu u(k') \bar{u}(k') \gamma_\nu u(k) = \frac{1}{2} \mathrm{tr} [\not{k} \gamma_\mu \not{k}' \gamma_\nu]$$

$$= 2 (k'_\mu k_\nu + k_\mu k'_\nu - g_{\mu\nu} \, k \cdot k')$$

Since $W^{\mu\nu}$ is the Compton amplitude, we must have $q^\mu W^{\mu\nu} = 0 = q^\nu W^{\mu\nu}$. This Ward identity simply states current conservation $\partial_\mu J^\mu = 0$.

The most general form of $W^{\mu\nu}$ consistent with this requirement is

$$W^{\mu\nu}(P,q) = \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}\right) W_1(P,q)$$

$$+ \left(P_\mu - q^\mu \frac{P.q}{q^2}\right)\left(P_\nu - q_\nu \frac{P.q}{q^2}\right) W_2(P,q).$$

The following to variables (due to Bjorken) are often used

$s =$

$$x = \frac{Q^2}{2P.q} = \frac{2E_k E_k'(1-\cos\theta)}{2m_p(E_k - E_{k'})}$$

$$y = \frac{2P.q}{2P.k} = \frac{E_k - E_{k'}}{E_k} \cong \frac{Q^2}{x}\frac{1}{s} \quad ; \quad s = (P+k)^2 = 2m_p E_k$$

Jacobian:

$$\begin{vmatrix} \partial x/\partial E_{k'} & \partial x/\partial\cos\theta \\ \partial y/\partial E_{k'} & \partial y/\partial\cos\theta \end{vmatrix} = \begin{vmatrix} \cdots & -2E_k E_{k'}/2m_p(E_k - E_{k'}) \\ -1/E_k & 0 \end{vmatrix}$$

$$= \frac{E_{k'}}{2m_p(E_k - E_{k'})} = \frac{2E_{k'}}{s\,y}$$

$$\int \frac{d^3k'}{2E_{k'}} = \int \frac{dE_{k'}}{2} E_{k'} \int_{-1}^{1} d\cos\theta \, 2\pi = \int dx \int dy \; y s \frac{2\pi}{4}$$

The cross section is thus

$$\frac{d^2\sigma}{dx dy} = \frac{1}{\cancel{2}\cancel{s}} \frac{e^4}{Q^4} \frac{1}{(2\pi)^3} \cancel{2} (k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu} k k') \frac{2\pi}{4} y \cancel{s}$$

$$\cdot \, 2 \, \mathrm{Im} \, W^{\mu\nu}$$

$$= \frac{2\alpha^2 y}{(Q^2)^2} (k_\mu k'_\nu + k'_\mu k_\nu - g_{\mu\nu} k \cdot k') \, \mathrm{Im} \, W^{\mu\nu}$$

Also the lepton tensor vanishes when contracted with q^μ, so

$$\frac{d^2\sigma}{dx dy} = \frac{2\alpha^2 y}{(Q^2)^2} \Big[2 k \cdot k' \, \mathrm{Im} \, W_1 +$$

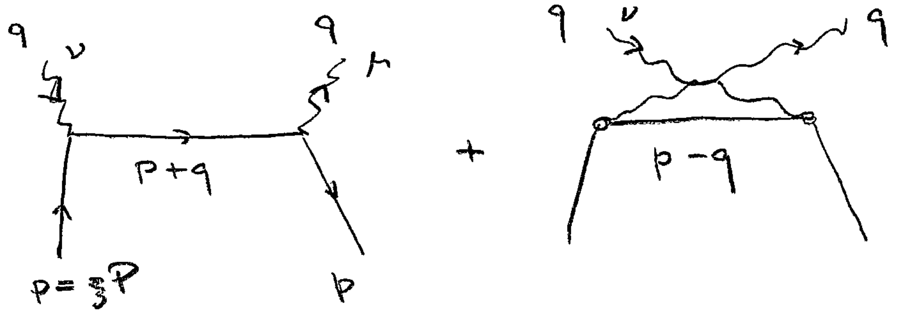
$$2 P \cdot k \, P k' \, \mathrm{Im} \, W_2 \Big]$$

$$= \frac{2\alpha^2 y}{(Q^2)^2} \left[Q^2 \, \mathrm{Im} \, W_1 + \frac{s^2}{2} (1-y) \, \mathrm{Im} W_2 \right]$$

$$\left[Q^2 = xys \right]$$

Before analyzing the hadronic matrix element, let's calculate $\hat{W}_{\mu\nu}$, the $W_{\mu\nu}$ for a quark instead of a proton state. Let's further write the quark momentum as $p = \xi P$. (we will later assume that the quark carries a fraction ξ of the proton momentum.)

$\hat{W}_{\mu\nu} =$



$$= i e_q^2 \, \frac{1}{2} \sum_s \bar{u}(p,s) \gamma^\mu \frac{i(\not{p}+\not{q})}{(p+q)^2 + i\varepsilon} \gamma^\nu u(p,s) + \begin{pmatrix} \mu \leftrightarrow \nu \\ q \leftrightarrow -q \end{pmatrix}$$

$$= -e_q^2 \, \frac{1}{2} \, \mathrm{tr}\left[\not{p} \gamma^\mu (\not{p}+\not{q}) \gamma^\nu\right] \frac{1}{2p\cdot q - Q^2 + i\varepsilon} + (\ldots)$$

Since $Q^2 > 0$, only the first diagram has an imaginary part.

$$\mathrm{Im}\left[\frac{1}{2p\cdot q - Q^2 + i\varepsilon}\right] = -\pi\,\delta(2\xi P\cdot q - Q^2) = -\frac{\pi}{ys}\,\delta(\xi - x)$$

$$\mathrm{Im}\, W^{\mu\nu} = \frac{\pi}{ys} e_q^2 \,\delta(\xi - x)\left[-g_{\mu\nu}\, ys\xi + 4\xi^2 P_\mu P_\nu + \ldots\right]$$

$$\mathrm{Im}\,\hat{W}_1 = \pi e_q^2\,\delta(\xi - x)\,\xi$$

$$\mathrm{Im}\,\hat{W}_2 = \frac{4\pi e_q^2}{ys}\,\xi^2\,\delta(\xi - x)$$

With this result at hand, we can now discuss the parton model. This model assumes that the proton is composed of partons (i.e. quarks and gluons). The partons carry fractions of the proton momentum. The proton scattering cross section is obtained by multiplying the probability $f_{q/p}(\xi)$ to find a quark with fraction ξ by the cross section for the quark to scatter.

Corrects the flux factor $\frac{1}{2s} \to \frac{1}{2\xi s}$

$$\text{So}: \quad \mathrm{Im}\,W_1 = \sum_q \int_0^1 \frac{d\xi}{\xi}\, f_{q/p}(\xi)\cdot \mathrm{Im}\,\hat{W}_1(\xi P, Q^2)$$

$$= \pi \sum_q e_q^2\, f_{q/p}(x)$$

$$\mathrm{Im}\,W_2 = \frac{4\pi e_q^2}{ys}\, x \sum_q e_q^2\, f_{q/p}(x)$$

The relation

$$\mathrm{Im}\, W_1 = \frac{ys}{4x}\, \mathrm{Im}\, W_2$$

is called the Callan-Gross relation.

It is characteristic for spin-$\frac{1}{2}$ partons. For spin 1, one obtains $W_2 = 0$. The fact that the relation was supported experimentally was taken as an indication that the quarks could be the partons (partons were introduced before QCD.)

Plugging into the formula for the cross section, one has

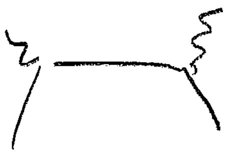
$$\frac{d\sigma}{dx\,dy} = \frac{2\pi\alpha^2 s}{Q^4}\left[1+(1-y)^2\right]\sum_q e_q^2\, x\, f_{q/p}(x)$$

While the x-dependence of the cross section is given by the a-priori unknown function $f_{q/p}(x)$, the Q^2-dependence is predicted.

The data indeed were compatible with the $1/Q^4$ scaling of the cross section supporting the parton model interpretation.

The $1/Q^4$ scaling is called Bjorken scaling. Higher orders in QCD lead to logarithmic corrections to the scaling law.

Since the hadronic matrix element is given by a product of currents, it is tempting to try to analyze it using the OPE. However the fact that the proton has a large energy ruins the expansion



$$\sim \frac{1}{2P\cdot q - Q^2} \overset{?}{=} \frac{1}{-Q^2}\left\{1 + \frac{2Pq}{Q^2} + \dots\right\}$$

Since P has large energy, we cannot neglect $2P\cdot q \sim Q^2$ and the expansion breaks down.

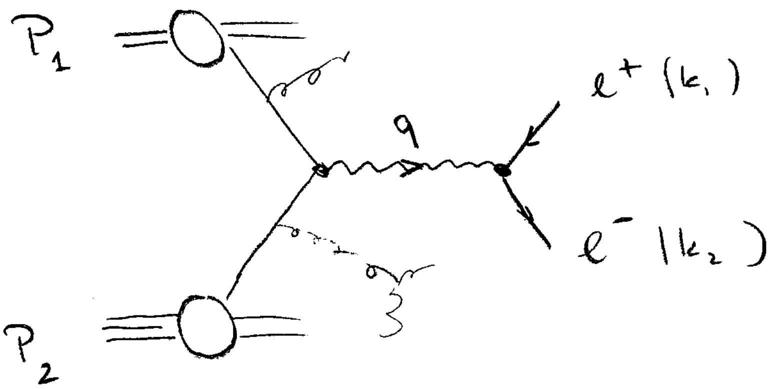
The traditional way of dealing with this problem is to expand anyway and then to resum all those higher-dim operators which are not suppressed. This is quite cumbersome.

Our effective theory was designed to analyze processes with energetic particles and will allow us to derive the result in a quite straightforward way.

9. Hadron - hadron collisions

9.1. Lepton pair production

One of the most basic processes at a hadron collider is Drell-Yan production, the reaction in which a lepton pair with high invariant mass M emerges after the collision, $pp \to e^+e^- + X$.



$$M^2 = q^2 = (k_1+k_2)^2$$

We work in the C.M.S. and parameterize

$$P_1^\mu = E(1,0,0,1) = E\,n^\mu ,$$
$$P_2^\mu = E(1,0,0,-1) = E\,\bar{n}^\mu .$$

The cross section

$$d\sigma = \frac{1}{2S} \sum_X |\mathcal{M}_{P_1 + P_2 \to k_1 + k_2 + p_X}|^2$$

$$\cdot \frac{d^3 k_1}{2E_1 (2\pi)^3} \frac{d^3 k_2}{2E_2 (2\pi)^3}$$

can be written as a leptonic times a hadronic part.

$$d\sigma = \frac{1}{2S} \frac{e^4}{M^4} \sum_{spins} \bar{u}(k_1) \gamma^\nu v(k_2) \bar{v}(k_2) \gamma^\mu u(k_2)$$

$$\times \frac{1}{4} \sum_{spins} \sum_X (2\pi)^4 \delta^{(4)}(P_1 + P_2 - q - q_X)$$

$$< P_1 P_2 | J^\mu(0) | X > < X | J^\nu(0) | P_1 P_2 >$$

$$= \frac{8\pi^2 \alpha^2}{S M^4} L^{\mu\nu} W_{\mu\nu} .$$

$$L^{\mu\nu} = \sum_{spins} \bar{u}(k_1) \gamma^\nu v(k_2) \bar{v}(k_2) \gamma^\mu u(k_1)$$

$$= 4 [k_1^\mu k_2^\nu + k_1^\nu k_2^\mu - k_1 \cdot k_2 g^{\mu\nu}]$$

$$W_{\mu\nu} = \frac{1}{4}\sum_{\text{spins}} \int d^4x\, e^{-iqx} \langle P_1 P_2 | J^\mu(x)\, J^\nu(0) | P_1 P_2 \rangle$$

$$= \frac{1}{4}\sum_{\text{spins}} \int d^4x\, e^{-iqx} \sum_X \langle P_1 P_2 | \overset{e^{i\hat{P}x} J^\mu(0) e^{-i\hat{P}x}}{J^\mu(x)} | X \rangle \langle X | J^\nu(0) | P_1 P_2 \rangle$$

$$= \frac{1}{4}\sum_{\text{spins}} \int d^4x\, e^{-iqx} \sum_X e^{i(P_1+P_2-P_X)} \langle P_1 P_2 | J^\mu(0) | X \rangle \langle X | J^\nu(0) | P_1 P_2 \rangle$$

$$= \frac{1}{4}\sum_{\text{spins}} (2\pi)^4\, \delta^{(4)}(P_1+P_2-P_X) \langle P_1 P_2 | J^\mu(0) | X \rangle \langle X | J^\nu(0) | P_1 P_2 \rangle$$

After integrating over the lepton angles, one obtains

$$\frac{d\sigma}{dM^2} = \frac{2\alpha^2}{3M^2 S}\, \frac{1}{4}\sum_{\text{spins}} \langle P_1 P_2 | W | P_1 P_2 \rangle$$

where

$$W(M) = -\int \frac{d^4q}{(2\pi)^3}\, \Theta(q^0)\, \delta(q^2 - M^2)$$

$$\times \int d^4x\, e^{-iqx} J^\mu(x) J_\mu(0) .$$

Now we'd like to analyze this process in the effective theory. We now have two directions with large energy flow and therefore two sets of collinear fields.

The building blocks for leading-power operators are

$$\left.\begin{matrix} \chi_1 \\ \mathcal{B}^\mu_{1\perp} \end{matrix}\right\} \begin{matrix} \text{collinear quark and gluon field} \\ \text{large energy in } n\text{-direction} \\ P \mathrel{\hat{=}} \bar{n}\cdot p\, \frac{n^\mu}{2} \sim E\, n^\mu \end{matrix}$$

$$\left.\begin{matrix} \chi_2 \\ \mathcal{B}^\mu_{2\perp} \end{matrix}\right\} \begin{matrix} \text{collinear fields in } \bar{n}^\mu \\ \text{direction} \end{matrix}$$

We would like to show that

$$W(M) = \int ds \int dt \; C_{qq}(s,t,M^2)\, \bar{\chi}_1(s\bar{n}) \frac{\not{\bar{n}}}{2} \chi_1(0)\, \bar{\chi}_2(tn) \frac{\not{n}}{2} \chi_2(0)$$

$$+ \int ds \int dt \; C_{qg}(s,t,M^2)\, \bar{\chi}_1(s\bar{n}) \frac{\not{\bar{n}}}{2} \chi_1(0)\, \mathrm{tr}\left[\mathcal{B}^\mu_{2\perp}(tn)\, \mathcal{B}^\perp_{2\mu}(0) \right]$$

$$+ \quad \text{"gluon-1 , quark-2"} \quad + \quad \text{"gluon-1 , gluon-2"}$$

If the operator has this form, then

1.) the soft fields will decouple like in DIS

2.) the proton matrix elements will factor into products of parton distribution functions.

Let us now argue that all non-vanishing contributions are of the above form. First of all, only operators with two fields from each sector have nonvanishing matrix elements, because we take color-singlet matrix elements

$$\langle P_1 | \chi(0) | P_1 \rangle = 0 \quad , \text{etc.}$$

Also, to be able to form a color-singlet we need two gluon fields or two quark fields from each sector.

Let us look at the operator built from four quark fields. It's general form is

$$\mathcal{O} = \bar{\chi}^i_{1,\alpha} \; \chi^j_{1,\beta} \; \bar{\chi}^k_{2,\gamma} \; \chi^l_{2,\delta} \; C^{ijkl}_{\alpha\beta\gamma\delta}$$

Here $\alpha, \beta, \gamma, \delta$ are Dirac indices, i, j, k, l are color indices.

A basis of Dirac matrices Γ_1^A relevant for $\bar{\chi}_1 \Gamma_1^A \chi_1$ is $\Gamma_1^A \in \{ \frac{\not{k}}{2}, \frac{\not{k}}{2}\gamma_5, \frac{\not{k}}{2}\gamma_\mu^\perp \}$. Because of the constraint $\not{k}\chi_1 = 0$, only four matrices are needed. The basis for χ_2 is $\Gamma_1^B \in \{ \frac{\not{k}}{2}, \frac{\not{k}}{2}\gamma_5, \frac{\not{k}}{2}\gamma_\mu^\perp \}$.

A general operator has the form

$$\mathcal{O} = \sum_{A,B} C_{AB} \; \bar{\chi}^i_1 \Gamma_1^A \chi^j_1 \; \bar{\chi}^k_2 \Gamma_2^B \chi^l_2 \, .$$

Because the operator W is a scalar, $C_{AB} \propto \delta_{AB}$. Furthermore, each fermion pair can be in a color singlet or a color octet.

The most general form of the four-fermion operator is

$$\mathcal{O} = C_S^1 \, \bar\chi_1 \frac{\not{\bar n}}{2} \chi_1 \; \bar\chi_2 \frac{\not n}{2} \chi_2$$

$$+ C_P^1 \, \bar\chi_1 \frac{\not{\bar n}}{2} \gamma_5 \chi_1 \; \bar\chi_2 \frac{\not n}{2} \gamma_5 \chi_2$$

$$+ C_V^1 \, \bar\chi_1 \frac{\not{\bar n}}{2} \gamma_\mu^\perp \chi_1 \; \bar\chi_2 \frac{\not n}{2} \gamma_\mu^\perp \chi_2$$

$$+ C_S^8 \, \bar\chi_1 \frac{\not{\bar n}}{2} t^a \chi_1 \; \bar\chi_2 \frac{\not n}{2} t^a \chi_2$$

$$+ C_P^8 (\dots) + C_V^8 (\dots)$$

After performing the decoupling transformation the color-singlet operators factorize

$$\mathcal{O} \longrightarrow C_S^1 \, \bar\chi_1^{(0)} \frac{\not{\bar n}}{2} \chi_1^{(0)} \; \bar\chi_2^{(0)} \frac{\not n}{2} \chi_2^{(0)} + C_P^1 (\dots) + C_V^1$$

$$+ C_S^8 \, \bar\chi_1^{(0)} S_n^\dagger(0) \frac{\not{\bar n}}{2} t^a S_n(0) \chi_1^{(0)} \; \bar\chi_2^{(0)} S_{\bar n}^\dagger(0) t^a S_{\bar n}(0) \frac{\not n}{2} \chi_2^{(0)}$$

$$+ \dots$$

but in the color-octet operators, the soft Wilson lines do <u>not</u> cancel.

We can write

$$S_n^+ t^a S_n = 2\,\mathrm{tr}\left[t^b S_n^+ t^a S_n \right] t^b$$

since $\{ 1\!\!1, t^b \}$ is a basis of 3×3 matrices. So

$$① = C_S^1 (\ldots) + C_P^1 (\ldots) + C_V^1$$

$$+ C_S^8 \cdot S_{cd}\, \bar{\chi}_1 t^c \frac{\not{\bar{n}}}{2} \chi_1\, \bar{\chi}_2 t^d \frac{\not{n}}{2} \chi_2$$

$$+ \ldots$$

$$S_{cd} = 2\,\mathrm{tr}\left[t^c S_n^+ t^a S_n \right] 2\,\mathrm{tr}\left[t^d S_{\bar{n}}^+ t^a S_{\bar{n}} \right]$$

$$\left(= 2\,\mathrm{tr}\left[t^c S_n^+ S_{\bar{n}} t^d S_{\bar{n}}^+ S_n \right] \right)$$

Taking the proton matrix element, all color-octet operators vanish, because the proton is a color singlet.

Of the singlet operators, only

$$\langle p_1 |\, \bar{\chi}_1 \frac{\not{\bar{n}}}{2} \chi_1 \,| p_1 \rangle$$

is nonvanishing.

The $\bar{\chi}_1 \frac{\not{n}}{2} \gamma_5 \chi_1$ matrix element vanishes because of parity conservation, the $\bar{\chi}_1 \frac{\not{n}}{2} \gamma_\perp^\mu \chi$ vanishes because it can only depend on the vectors $n^\mu, \bar{n}^\mu$ and P_1^μ and none of those has a $\perp$-component.

So, for the four-fermion case, we have shown that

$$W = \int ds \int dt \, C_{qq}(s, t, M^2) \, \bar{\chi}_1(s\bar{n}) \frac{\not{\bar{n}}}{2} \chi_1(0) \, \bar{\chi}_2(tn) \frac{\not{n}}{2} \chi_2(0)$$

+ "operators which have vanishing proton matrix elements"

+ "operators with $\mathcal{B}_\perp$ fields"

Taking the quark matrix element, gives

$$\hat{W}_{qq} = \tilde{C}(\bar{n}\cdot p_1, n\cdot p_2, M^2) \, \bar{n}\cdot p_1 \, n\cdot p_2$$

and the proton matrix element gives the usual PDFs. After extending the discussion to the other operators, we get the result for W:

$$\frac{1}{4}\sum_{\text{spin}} \langle P_1 P_2 | W | P_1 P_2 \rangle$$

$$= \sum_{i,j} \int_{-1}^{1} \frac{d\xi_1}{\xi_1} \int_{-1}^{1} \frac{d\xi_2}{\xi_2} \; \hat{W}_{ij}(\xi_1, \xi_2, M^2)\, f_i(\xi_1) f_j(\xi_2)$$

where $i,j = g, u, d, s, \ldots$

As for DIS, the hadronic cross section is obtained by convoluting partonic cross sections with the PDFs.

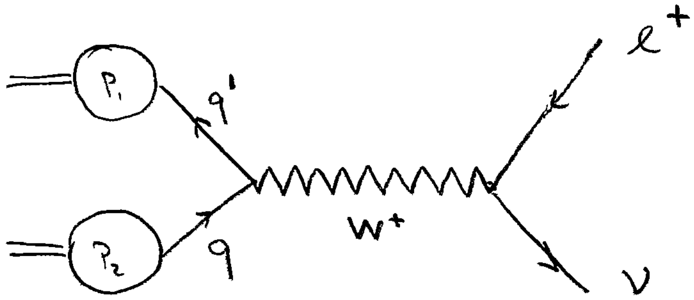
We have analyzed $d\sigma/dM^2$, but our analysis would work equally well for quantities which are more differential in the lepton momentum, such as $\dfrac{d\sigma}{dM^2\, dy\, dq_T}$.

Here y is the photon rapidity $y = \frac{1}{2} \ln\left(\dfrac{q^0 + q^z}{q^0 - q^z}\right)$ and $q_T = \sqrt{q_x^2 + q_y^2}$. Another common variable is the "transverse mass" $m_T^2 = M^2 + q_x^2 + q_y^2$

9.11.

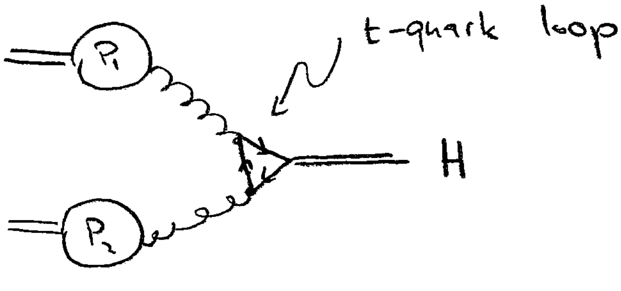
At higher energy, one needs to include both the photon and the Z-boson, to obtain the correct result for the cross section. (see figures)

A closely related process is W-production



the cross section is proportional to $|V_{qq'}|^2$.

Also Higgs production is rather similar



The Higgs boson coupling $\propto m_f$. The top contribution accounts for 95% of the signal,

9.2. Jet production

For the case of DIS and for Drell-Yan, we were able to show that the hadronic cross section is obtained by convoluting the partonic cross section with the parton distribution functions.

Although a formal proof is lacking*, it is generally believed that the same factorization theorem is also valid for more exclusive processes, which involve jets in the final state so that

$$d\sigma(N_1(P_1)+N_2(P_2)\longrightarrow n \text{ jets}) =$$

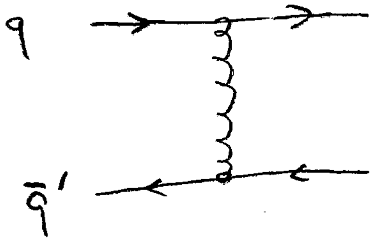
$$\int d\xi_1 \int d\xi_2 \sum_{i,j} d\hat{\sigma}(i(\xi_1 P_1)\, j(\xi_2 P_2)\longrightarrow n \text{ jets}) \times f_{i/N_1}(\xi_1)\, f_{j/N_2}(\xi_2)$$

---

* See arXiv: 0808.2191 for a discussion of jet-production in SCET.

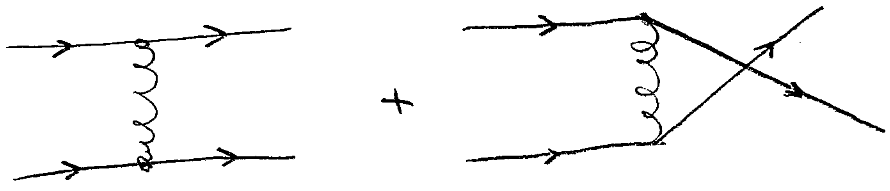
As an example, let's consider two-jet production at leading power in α_s. Even in this simple example, a significant number of diagrams contribute. Let's draw those channel by channel.

a.) For $i = q$, $j = \bar{q}'$, $q \neq q'$

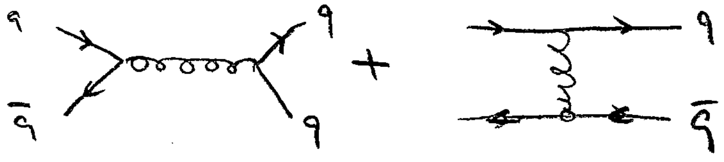


b.) $i = q$, $j = q'$. Same as a.)

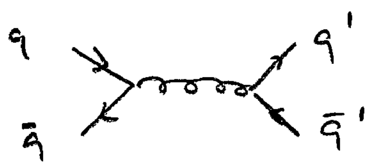
c.) $i = q$, $j = q$



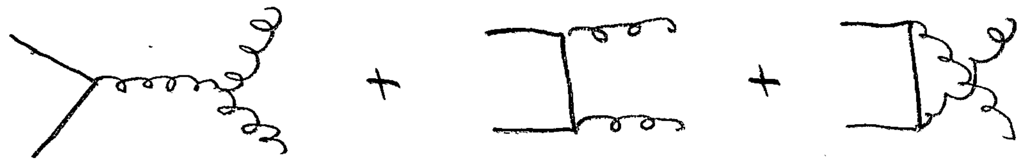
d.) $i = q$, $j = \bar{q}$



for $q\bar{q}$ final state



for $q\bar{q}'$ final state

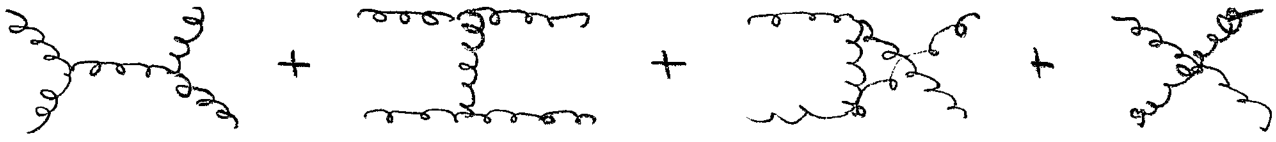


for gg

Note that the diagrams with the same final state interfere, but not the other ones.

$$d\hat{\sigma}_{q\bar q} \sim |\mathcal{M}_{q\bar q \to q\bar q}|^2 + |\mathcal{M}_{q\bar q \to q'\bar q'}|^2$$
$$+ |\mathcal{M}_{q\bar q \to gg}|^2 .$$

e.) $i = g$, $j = g$



for gg final state.

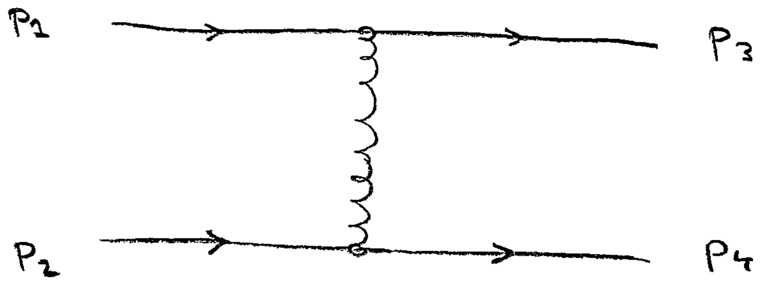
see d.) for the $q\bar q$ final state.

f.) $qg \to qg$ and $\bar{q}g \to \bar{q}g$

rotate the $\bar{q}q \to gg$ diagrams

To get the two-jet cross section, we need to apply a jet algorithm to the final state. However, because the final state only contains two particles we'll end up with two jets (unless our cone is so large that we absorb the entire final state into it.)

There is no point in trying to evaluate all these diagrams in the lecture, but let us calculate one of them to illustrate how to deal with the color structure, etc.



define $\hat{s} = (p_1+p_2)^2 = (\xi_1 P_1 + \xi_2 P_2)^2 = \xi_1 \xi_2 S$

$\hat{t} = (p_1 - p_3)^2$

$\hat{u} = (p_1 - p_4)^2 \qquad s+t+u=0.$

(we are neglecting the small quark masses)

$$im = (ig)^2 \frac{-i}{(p_1-p_3)^2} \bar{u}(p_3)\gamma^\mu t^a u(p_1)\, \bar{u}(p_4)\gamma^\mu t^a u(p_2)$$

To get the cross section, we sum over colors and spins of the out-going particles and average over the incoming.

$$d\sigma \propto \frac{1}{4}\frac{1}{N_c^2}\sum_{\substack{\text{spins}\\ \text{colors}}} |m|^2$$

$$= \frac{g^4}{4N_c^2}\sum_{\substack{\text{spins}\\ \text{colors}}} \bar{u}(p_3)\gamma^\mu t^a u(p_1)\, \bar{u}(p_1)\gamma^\nu t^b u(p_3)$$
$$* \; \bar{u}(p_4)\gamma^\mu t^a u(p_2)\, \bar{u}(p_2)\gamma^\nu t^b u(p_4))$$

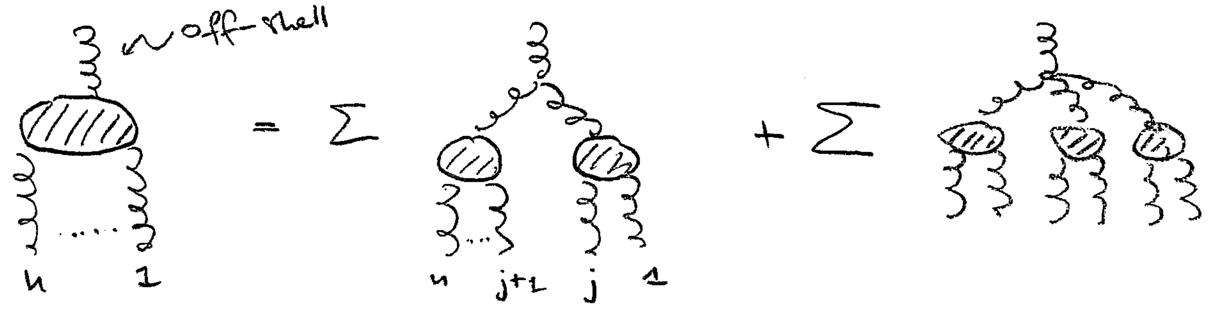
$$= \frac{g^4}{4N_c^2}\frac{1}{\hat{t}^2} tr[\not{p}_3\gamma^\mu \not{p}_1 \gamma^\nu]\, tr[\not{p}_4 \gamma^\mu \not{p}_2 \gamma^\nu]$$
$$\cdot\; tr_c[t^a t^b]\, tr_c[t^a t^b]$$

$$= \frac{g^4}{4N_c^2}\frac{1}{\hat{t}^2}\, 8(\hat{s}^2+\hat{u}^2)\, \underbrace{\frac{1}{2}\delta^{ab}\frac{1}{2}\delta^{ab}}_{\frac{N^2-1}{4}}.$$

While the computation of these diagrams is straightforward, it quickly becomes quite tedious. Also, for processes with more legs, the number of diagrams quickly increases, e.g. for $gg \to ng$, we have

n	2	3	4	5	6	7	8
diagrams	4	25	220	2485	34300	559'405	10'525'900

For this reason automated codes were developed which generate and calculate the corresponding Feynman diagrams and then integrate them numerically over phase-space. For large n, Feynman diagrams become inefficient and one uses recursion relations, which allow one to obtain amplitudes with more legs by adding a leg to lower-n amplitudes, e.g.



For a review, see 0707.3342 by S. Weinzierl. While these off-shell recursion relations are known since a long time, recently new types of on-shell recursion relations were developed.*

Two general-purpose and user friendly tree-level codes are CompHEP (http://comphep.sinp.msu.ru/) and MadGraph (http://madgraph.hep.uiuc.edu).

see slides for a calculation of pp -> 2 jets using MadGraph, MadEvent and MadAnalysis.

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* Off-shell: Berends & Giele '88

On-shell: Britto, Cachazo, Feng, Witten '04

Cachazo, Svrcek, Witten '04

10. Parton shower and Monte Carlo Methods

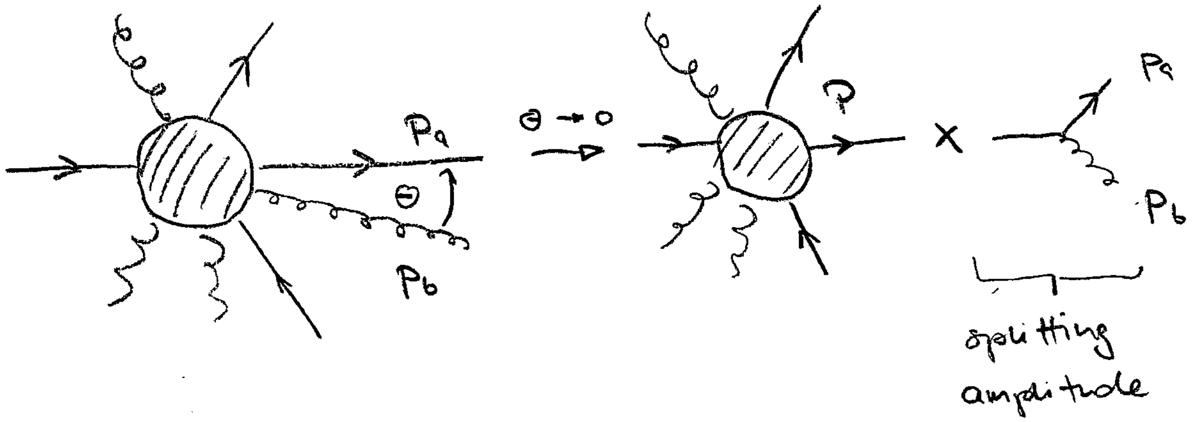
In our discussion of jet-production, we have seen that the complexity of perturbative calculations rapidly explodes as the number of external particles increases. Also, our ability to calculate higher-order corrections is quite limited. Only this year, the first NLO results for 2 → 4 processes were obtained and at NNLO only inclusive hadron collider cross sections (Drell-Yan, Higgs production) are known.

There are two limits, where higher order emissions take a simple form: the emissions factorize in the collinear or soft limit. In the collinear limit one finds

$$d\sigma_{n+1} = d\sigma_n \cdot \overset{\text{splitting function}}{\overbrace{dP}}$$

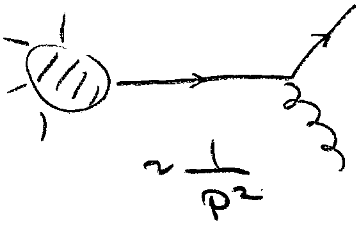
$$d\sigma_{n+1}(p_1, p_2, \dots, p_n, p_{n+1}) = d\sigma_n(p_1, p_2, \dots, p_{n-1}, \overset{P = p_n + p_{n+1}}{\overset{\parallel}{P}}) \cdot dP(p_n, p_{n+1})$$

Or graphically

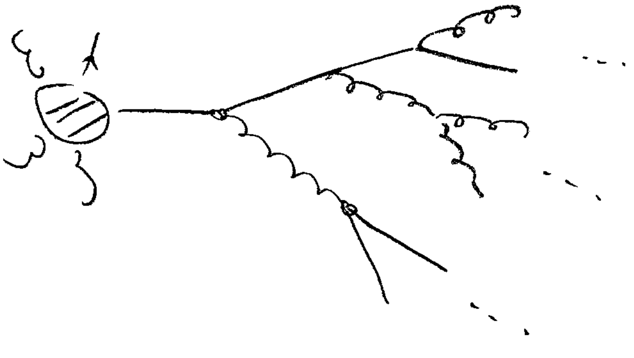


By iterating this relation, one can generate collinear emissions of arbitrary high order.

At the same time, these emissions are enhanced because the propagator denominator $p^2 = (p_a + p_b)^2 \to 0$ as $\theta \to 0$.



A parton shower generates these collinear emissions via a Monte-Carlo process.



Such Monte-Carlo (MC) programs, or event generators, are widely used to analyze collider processes.

We will first derive the splitting functions and then show how the factorization can be used to build a MC event generator.

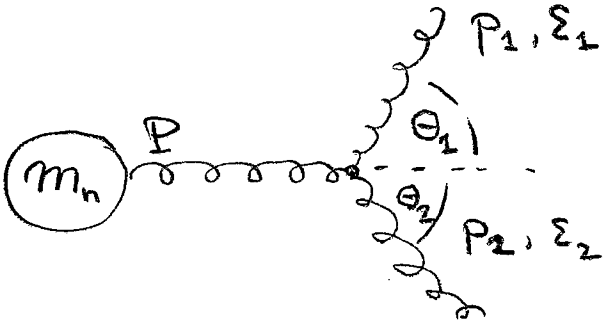
10.1. Parton branching and splitting functions

Let's look at the $g \to gg$ splitting first

$P = (E, 0, 0, p)$

$p_1 = (E_1, \sin\Theta_1 E_1, 0, \cos\Theta_2 E_2)$

$p_2 = (E_2, -\sin\Theta_2 E_2, 0, \cos\Theta_2 E_2)$



Write $E = z E$

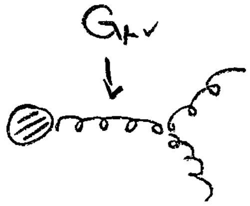
$E = (1-z) E$

$P^2 = (p_1+p_2)^2 = 2E_1E_2(1-\cos\Theta) \cong z(1-z)\Theta^2 E^2$

$E_1 \sin\Theta_1 = E_2 \sin\Theta_2 \Rightarrow z\Theta_1 \cong (1-z)\Theta_2$

$\Theta = \Theta_1 + \Theta_2 = \Theta_1 + \frac{z}{1-z}\Theta_2 = \frac{\theta_1}{1-z} = \frac{\Theta_2}{z}$

Internally we have a gluon propagator



$$G_{\mu\nu}(p) = -\frac{i g_{\mu\nu}}{p^2} + \xi \frac{i\, p_\mu p_\nu}{p^4}$$

$$\begin{aligned} g_{\mu\nu} &= \sum_{\lambda=0}^{3} \varepsilon_\mu(\lambda)\, \varepsilon^*_\nu(\lambda) \\ &= \varepsilon_+^\mu \varepsilon_-^{*\nu} + \varepsilon_-^\mu \varepsilon_+^{*\nu} - \sum_{i=1}^{2} \varepsilon_T^\mu \varepsilon_T^{*\nu} \end{aligned}$$

$$\left.\begin{array}{l} \varepsilon_+^\mu \propto n^\mu \sim p^\mu \\ \varepsilon_-^\mu \propto \bar{n}^\mu \end{array}\right\} \text{ are the unphysical polarizations.}$$

The amputated Green's functions fulfill $p_\mu \Gamma^{\mu\nu\ldots} = 0$ (Ward identity) so that only the physical polarizations give a nonzero contibtions, so that we can replace

$$G_{\mu\nu}(p) \longrightarrow \frac{i \sum_{\lambda=1}^{2} \varepsilon_T^\mu(\lambda)\, \varepsilon_T^{*\nu}(\lambda)}{p^2}$$

for the internal propagator.

Let us now evaluate the splitting amplitude for the case, where the internal polarization vector is ε_P.

$p_1 + p_2 = P$

$$V_{ggg} = g_s f_{abc}\, \varepsilon_P^\mu \varepsilon_1^\nu \varepsilon_2^\rho$$

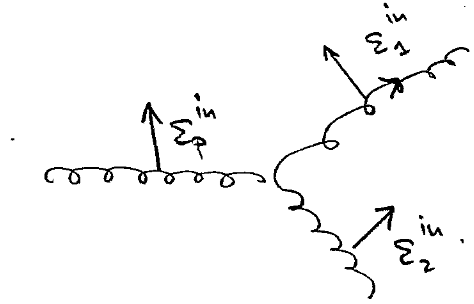
$$\times \Big(g_{\mu\nu} \overset{2p_1^\rho + p_2^\rho}{(P \underset{\sim}{+} p_1)_\rho} + g_{\nu\rho} \overset{-P^\mu + 2p_2^\mu}{(-p_1 + p_2)_\mu} + g_{\rho\mu} \overset{-2p_2 - p_1}{(-p_2 - P)_\nu} \Big)$$

$$= g_s f_{abc} \Big[\varepsilon_P \cdot \varepsilon_1\, \overset{\checkmark}{2} p_1 \cdot \varepsilon_2 + \varepsilon_1 \overset{\checkmark}{\cdot} \varepsilon_2\, 2\varepsilon_P \cdot p_2$$

$$\varepsilon_P \cdot \varepsilon_2 (-2 p_2 \cdot \varepsilon_1) \Big]$$

we have used $P = p_1 + p_2$; $\varepsilon_i \cdot p_i = 0$.

To evaluate the expression further, let's choose a basis of polarization vectors. It is most natural to choose one of the polarization vectors ε_i^{in} to lie in the plane of the reaction and the other one out of the plane:



The other one, ε_i^{out} is chosen perpendicular to the plane.

Explicitly:

$$\varepsilon_p^{out} = \varepsilon_1^{out} = \varepsilon_2^{out} = (0,0,1,0)$$

$$\varepsilon_p^{in} = (0,1,0,0)$$

$$\varepsilon_1^{in} = (0, \cos\theta_1, 0, -\sin\theta_1) \cong (0,1,0,-\theta(1-z))$$

$$\varepsilon_2^{in} = (0, \cos\theta_2, 0, \sin\theta_2) \cong (0,1,0,\theta z)$$

Choosing all polarizations in the plane, one obtains

$$V_{ggg} = 2g_s f_{abc}\Big[-Ez\theta + E(1-z)z\theta$$
$$- E(1-z)\theta\Big]$$

$$= -2g_s f_{abc} E\theta(1-z+z^2)$$

We need to multiply by $\frac{i}{p^2}$ from the propagator denominator and square

$$\frac{1}{N^2-1}\sum_{\text{colors}}\left|\frac{1}{p^2}V_{ggg}\right|^2 = \frac{\overset{C_A(N^2-1)}{f_{abc}f_{abc}}}{N^2-1}\;\frac{4g_s^2}{p^2}\;\underbrace{\frac{E^2\theta^2}{p^2}}_{\frac{1}{z(1-z)}}(1-z+z^2)^2$$

$$= \frac{4C_A g_s^2}{p^2}\left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right] = \frac{4C_A g_s^2}{p^2}\,F(in,in,in)$$

Repeating the calculations for the other polarizations, we have

P	1	2	F
in	in	in	$\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)$
in	out	out	$z(1-z)$
out	in	out	$(1-z)/z$
out	out	in	$z/(1-z)$
all other combinations			0

Averaging over incoming and summing over outgoing polarizations, one finds

$$C_A \langle F \rangle = \hat{P}_{gg} = C_A \left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right]$$

The same splitting function governs the evolution of the gluon PDF.

The singularities at $z=0$ and $z=1$ correspond to soft emissions. In this case one of the gluons carries almost no energy. The singularities only arise for soft gluons with polarizations in the plane.

Instead of ε_p^{in} and ε_p^{out}, let us consider a polarization vector $\varepsilon_\phi = \cos\phi\, \varepsilon_p^{in} + \sin\phi\, \varepsilon_p^{out}$. In this case one gets

$$F_\phi = \overbrace{\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)}^{\text{unpolarized result.}} + z(1-z)\cos 2\phi$$

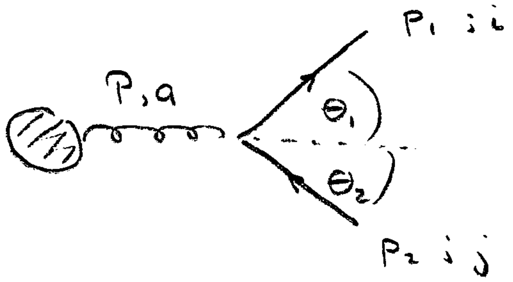
So the branching is enhanced in the plane of the polarization vector, but the dependence is weak:

$$F_\phi = 2{,}25 + 0{,}25\cos 2\phi \quad \text{for } z = \tfrac{1}{2}\,.$$

In addition to the $g \to gg$ branching, one also needs the $g \to q\bar{q}$ and the $q \to qg$ branchings.

For $g \to q\bar{q}$, one has

$$V_{q\bar{q}g} = ig(t^a)_{ij}\, \bar{u}(p_1)\gamma_\mu v(p_2)\varepsilon_p^\mu$$



To evaluate the expression, one needs to use explicit forms of the spinors, expanded for small angles. In the Dirac basis

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix} \; ; \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

the spinors are

$$\frac{u_+(p_1)}{\sqrt{E_1}} = \begin{pmatrix} 1 \\ \theta_1/2 \\ 1 \\ \theta_1/2 \end{pmatrix} \qquad \frac{u_-(p_1)}{\sqrt{E_1}} = \begin{pmatrix} \theta_1/2 \\ -1 \\ \theta_1/2 \\ -1 \end{pmatrix}$$

$$\frac{v_+(p_2)}{\sqrt{E_2}} = i\begin{pmatrix} -\theta_2/2 \\ -1 \\ \theta_2/2 \\ 1 \end{pmatrix} \qquad \frac{v_-(p_2)}{\sqrt{E_2}} = i\begin{pmatrix} -1 \\ \theta_2/2 \\ -1 \\ \theta_2/2 \end{pmatrix}$$

After plugging this into the matrix element, one has

$$|\mathcal{M}_{n+1}|^2 = \frac{2 g_s^2}{p^2} \overset{\frac{1}{2}}{\overset{\shortparallel}{T_F}} \mp |\mathcal{M}_n|^2$$

P	1	2	$\mp$
in	$\pm$	$\mp$	$\frac{1}{2}(1-2z)^2$
out	$\pm$	$\mp$	$1/2$

Note that there are no ~~soft~~ singularities ($z \to 0$, or $z \to 1$).

They only appear in gluon emissions.

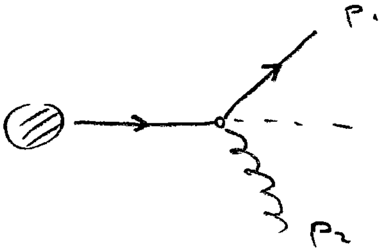
Averaged splitting function:

$$\hat{P}_{qg}(z) = T_F \langle F \rangle = T_F[z^2 + (1-z^2)]$$

$$F_\phi = z^2 + (1-z^2) - 2z(1-z)\cos(2\phi)$$

Finally, for $q \to qg$, one obtains

$$|\mathcal{M}_{n+1}|^2 = \frac{2g_s^2}{p^2} C_F F |\mathcal{M}_n|^2$$



P	1	2	
±	±	in	$\frac{1}{2}(1+z)^2/(1-z)$
±	±	out	$\frac{1}{2}(1-z)$

Averaged splitting function

$$\hat{P}_{qq}(z) = C_F \langle F \rangle = C_F \frac{1+z^2}{1-z}$$

$$F_\phi = \frac{1+z^2}{1-z} + \frac{2z}{1-z}\cos 2\phi$$

We have derived results for the n+1 particle amplitude squared $|\mathcal{M}_{n+1}|^2$ in terms of $|\mathcal{M}_n|^2$.

To get a relation for the cross sections, we also need to relate the phase-space in the two cases.

To do so, remember that

$$\int d^4p\, \Theta(p^0)\, \delta(p^2 - M^2) = \int \frac{d^3p}{2E}$$

therefore

$$\int dM^2 \int d^4p\, \Theta(p^0)\, \delta(p^2 - M^2)$$

$$= \underline{\int d^4p\, \Theta(p^0) = \int dM^2 \int \frac{d^3p}{2E}}$$

Now rewrite the (n+1)-particle phase-space

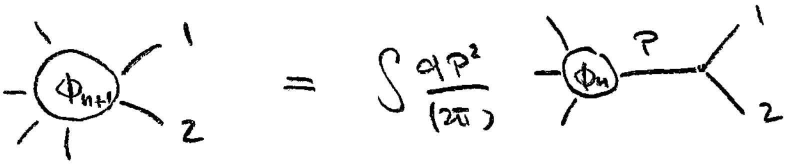
$$\int d\Phi_{n+1} = \int d\Phi_{n-1} \int \frac{d^3p_1}{2E_1 (2\pi)^3} \int \frac{d^3p_2}{2E_2 (2\pi)^3}$$

$$= \int d\Phi_{n-1} \int d^4P\, \Theta(P^0) \int \frac{d^3p_1}{2E_1 (2\pi)^3} \int \frac{d^3p_2}{2E_2 (2\pi)^3}\, \delta^{(4)}(P - p_1 - p_2)$$

$$= \int \frac{\overset{\overset{M2}{\downarrow}}{dP^2}}{(2\pi)} \int d\phi_{n-1} \int \frac{d^3P}{2E(2\pi)^3} \int \frac{d^3p_1}{2E_1(2\pi)^3} \int \frac{d^3p_2}{2E_2(2\pi)^3} (2\pi)^4 \delta(P-p_1-p_2)$$

$$= \int \frac{dP^2}{(2\pi)} \int d\underline{\underline{\Phi_n}} \cdot \int \frac{d^3p_1}{2E_1(2\pi)^3} \int \frac{d^3 p_2}{2E_2(2\pi)^3} (2\pi)^4 \delta(P-p_1-p_2)$$

Pictorially:



Now we want to approximate the integration over the two-particle phase-space in the limit where $\Theta \to 0$.

$$\int dP^2 \int \frac{d^3p_1}{2E_1} \int \frac{d^3p_2}{2E_2} \delta^4(P-p_1-p_2)$$

$$= \int dP^2 \int \frac{d^3p_1}{2E_1} \int d^4p_2\, \Theta(p_2^0)\, \delta(p_2^2)\, \delta^{(4)}(P-p_1-p_2)$$

$$= \int dP^2 \int \frac{d^3p_1}{2E_1} \Theta(E-E_1)\, \delta((P-p_1)^2)$$

$$= \int dP^2 \int dE_1 \frac{E_1}{2} \int \cos\theta_1 \int d\varphi\; \Theta(E-E_1)\, \delta\left[P^2 - 2EE_1(1-\cos\theta_1)\right]$$

$$\int d\theta \sin\theta \approx \frac{1}{2}\int d\theta^2$$

$$1-\cos\theta_1 \approx \frac{\theta_1^2}{2}$$

$$\downarrow$$

$$= \int dp^2 \int_0^1 dz \; \frac{zE^2}{2} \; \frac{1}{2}\int d\theta_1^2 \int d\varphi \; \delta\left[p^2 - E^2 z \,\theta_1^2\right]$$

$$= \int dp^2 \int_0^1 dz \; \frac{1}{4} \int d\varphi$$

So we have

$$\int d\Phi_{n+1} = \frac{1}{4(2\pi)^3} \int_0^\infty dp^2 \int_0^1 dz \int_0^{2\pi} d\varphi \int d\Phi_n$$

For the differential cross section, we get

$$d\sigma_{n+1} = d\sigma_n \; \frac{dp^2}{p^2} \; \frac{d\varphi}{(2\pi)} \; \frac{\alpha}{2\pi} \; C \cdot F \; dz$$

⌈Note: for $g \to gg$, the $d\sigma_{n+1}$ contains a factor $\frac{1}{2}$ for identical particles.⌋

If we don't care about the φ-dependence, we can average

$$d\sigma_{n+1} = d\sigma_n \; \frac{dp^2}{p^2} \; \frac{\alpha}{2\pi} \; \hat{P}(z)\, dz$$

10.2. Parton shower

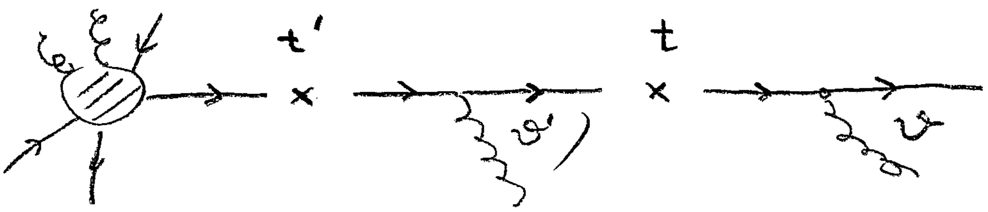
In the last section we have derived a relation between the cross section for n+1 partons in terms of the n-parton cross section in the collinear limit:

$$d\sigma_{n+1} = d\sigma_n \frac{\alpha}{2\pi} \frac{dt}{t} dz \, P_{i\to jk}(z) \frac{d\phi}{2\pi} \quad ; t = p^2$$

Here, $P_{i\to jk}$ is the splitting function for the $i\to jk$ branching ($q\to qg$, $g\to q\bar{q}$, $g\to gg$, $\bar{q}\to\bar{q}g$).

The splitting depends weakly on ϕ and also on the polarizations. For simplicity we'll suppress this dependence.

We can obtain the cross section for multiple emissions by iterating the above relation:



+ " gluon branchings "

$$d\sigma_{n+2} = \sum_{x,y} d\sigma_n \frac{\alpha(t')}{2\pi} \frac{dt'}{t'} P_x(z')\, dz' \frac{d\phi'}{2\pi}$$

$$\times \frac{\alpha(t)}{2\pi} \frac{dt}{t} P_y(z)\, dz \frac{d\phi'}{2\pi} \quad \Theta(t'-t)$$

The sum over x,y indicates that one should sum over all splittings which lead to the same final state.

Unfortunately, the results are not yet in useable form. If we would want to calculate a jet cross section, simply adding our tree-level results

$$\sigma_n^{tree} + \sigma_{n+1}^{tree} + \sigma_{n+2}^{tree} + \dots$$

gives the wrong result. At the same order in α_s as σ_{n+1}^{tree} also σ_n^{1-loop} will contribute, etc.

However, using unitarity, one can obtain also the virtual corrections at the same level of accuracy.

The amplitudes squared $|M|^2$ are probabilities for certain processes to happen. The splitting functions can therefore be viewed as probabilities for an additional emission

$$dP_{emis}(t+dt,t) = \frac{dt}{t}\frac{\alpha(t)}{2\pi}\int dz\, P_{i\to jk}(z)$$

So the probability <u>not</u> to have an emission is

$$dP_{no\ emis}(t+dt,t) = 1 - \sum_{jk} dP_{emis}$$

$$= 1 - \sum_{j,k}\frac{dt}{t}\frac{\alpha(t)}{2\pi}\int dz\,\hat{P}_{i\to jk}(z)$$

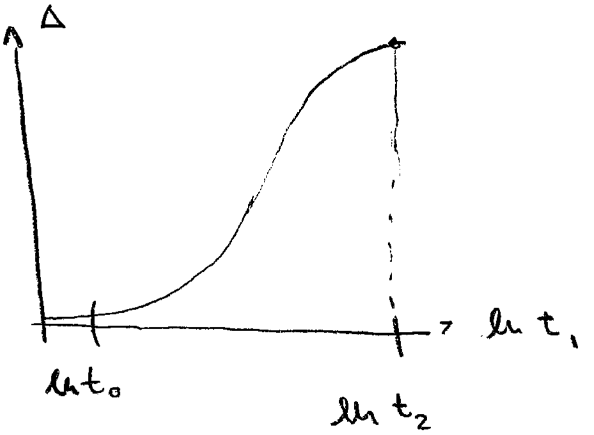
If we divide an interval $[t_1, t_2]$ in N small intervals $dt = (t_2-t_1)/N$, then

$$P_{no\ emis}(t_2,t_1) = \lim_{N\to\infty}\prod_{n=1}^{N}\left[1 - \sum_{jk}\frac{dt}{t_n}\frac{\alpha(t_n)}{2\pi}\int dz\,\hat{P}(z)\right]^N$$

$$\to \exp\left\{-\int_{t_1}^{t_2}\frac{dt}{t}\frac{\alpha(t)}{2\pi}\int dz\sum_{ke}\hat{P}_{i\to k,e}(z)\right\} \equiv \Delta_i(t_2,t_1)$$

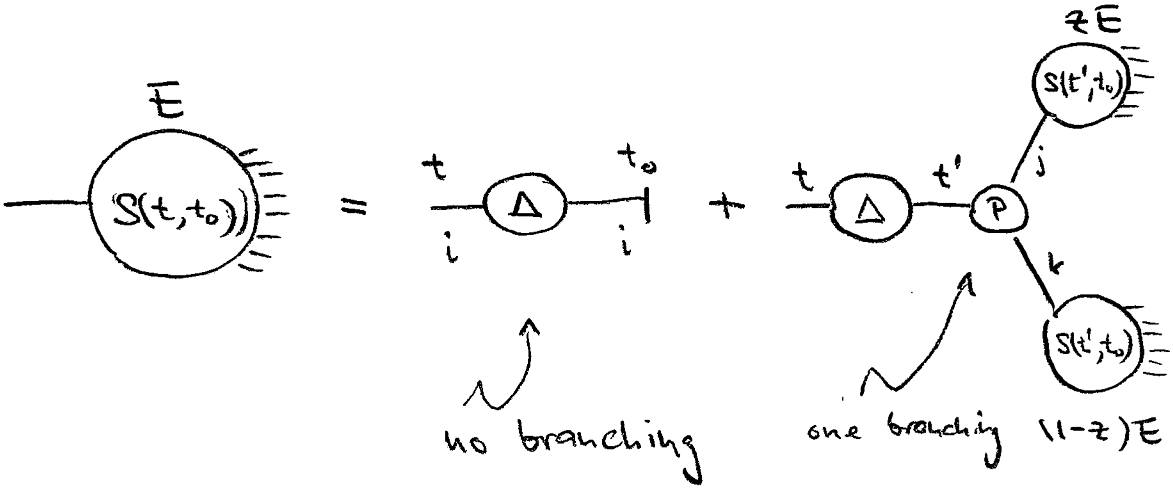
$\Delta_i(t_2,t_1)$ is called the Sudakov form factor.

Very roughly $\Delta(t_2, t_1) \sim \exp\left[-\frac{\alpha}{2\pi}\ln^2\left(\frac{t_2}{t_1}\right)\right]$



The probability for a parton with virtuality t_2 to not shower down to a small scale t_0 is very small

Now we can formulate the parton shower S



This form is now implemented into a computer code which will generate emissions according to probability, and terminate once it reaches t_0.

Let us discuss the computer implementation in some detail.

If we start from a given value of t, the probability to have an emission at t' is

$$dP = \Delta_i(t,t')\,\frac{\alpha(t')}{2\pi}\,\frac{dt'}{t'}\int\sum_{(jk)} P_{i\to jk}(t)\,dz\,\frac{d\varphi}{2\pi}$$

$$= d\Delta(t,t')\left[= \frac{d\Delta(t,t')}{dt'}\,dt'\right]$$

We would like our computer code to generate random values t' for the next branching distributed as dP, starting with a uniform random variable r in $[0,1]$

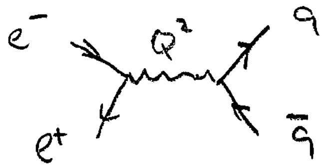
$$dP = f(t')\,dt' = 1\,dR \quad , \text{with} \quad f(t')dt' = dF(t')$$

$$\int_{t'}^{t} dt''\,f(t'') = \Delta(t,t') = \int_0^r 1\,dR = r$$

→ To get the value of t' generate random value $r \in [0,1]$ and solve $\Delta(t,t') = r$ for t' numerically.

Now we are in a position to formulate our shower algorithm:

1.) For each final-state colored parton, generate a shower with $t = Q^2$ (where Q^2 is a typical high scale of the process).



2.) For each shower: generate random number $0 < r < 1$.

Solve $r = \Delta_i(t, t')$ for t'

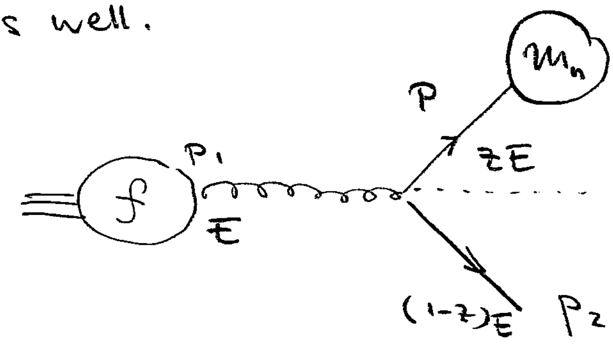
2a.) If $t' < t_0$ (cut-off) stop the shower

2b.) If $t' \geq t_0$ generate z, jk with probability $P_{i,jk}(z)$ and $0 < \varphi < 2\pi$ uniformly.

Assign $E_j = z E_i$; $E_k = (1-z) E_i$

3.) Restart shower for j and k, setting $t = t'$.

At a hadron collider, we also have incoming color-charged partons and would like to shower them as well.



Despite the changed kinematics, the basic formula

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha}{2\pi} P_{i \to jk}$$

is exactly the same as for the outgoing legs.

Note that additional emissions increase the value of t ~~for~~ for radiation off incoming legs. When implementing initial state showering, it is most efficient to evolve also in this case from large t to smaller t. The probability for this backward evolution is obtained by solving

$$\frac{f_i(\xi, \mu^2 = t')\, \Delta(t, t')}{f_i(\xi, \mu^2 = t)} = r$$

where the parton i carries the momentum fraction $\xi = \frac{E_i}{E_h}$. For an explanation, see Sjöstrand PLB 157:321, 1985.

Our shower evolves from large virtuality $t = P^2$ to lower values, but there are other possible choices such as the opening angle $t = E^2\Theta^2$, or the transverse momentum $t = p_T^2$. They are related as follows

$$P^2 = z(1-z)E^2\, 2(1-\cos\Theta) \cong z(1-z)\Theta^2 E^2$$

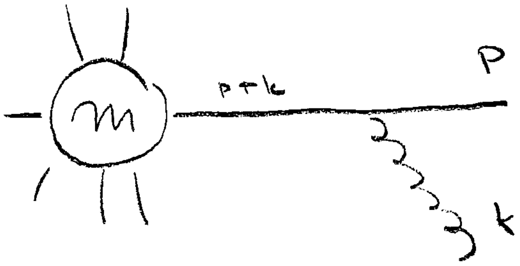
$$p_T^2 = E_1^2 \sin^2\Theta_1 \cong z^2\Theta_1^2 E^2 = z^2(1-z)^2\,\Theta^2 E^2$$

$$z^2 < 1$$

Note that $z(1-z) > \frac{P^2}{E^2}$ so that $z \in [\frac{t}{E^2}, 1 - \frac{t}{E^2}]$ for a virtuality ordered shower and $z \in [\frac{\sqrt{t}}{E}, 1 - \frac{\sqrt{t}}{E}]$ for a p_T^2-ordered shower. In the following, we'll see that only angular ordering gives correct results also for soft emissions. Nevertheless, Pythia, the most popular MC program, uses p_T-ordering (and used to use virtuality ordering).

10.3. Soft emissions and coherent branching.

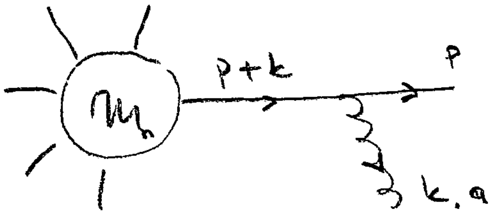
So far, we discussed the calculation of higher-order corrections from collinear emissions.



$$\frac{1}{(p+k)^2} = \frac{1}{2pk} = \frac{1}{2\omega E(1-\cos\Theta)}$$

These are enhanced, because the intermediate propagator goes on-shell for $\Theta \to 0$. However, the same enhancement also appears in the soft limit $\omega \to 0$. We have encountered this soft enhancement as $P(z) \sim \frac{1}{z}$ $z \to 0$ [or $P(z) \sim \frac{1}{1-z}$ for $z \to 1$] when we evaluated the splitting functions. Since both enhancements are logarithmic $\int_{z_0}^{z_1} dz \, \frac{1}{z} = \ln\left(\frac{z_1}{z_0}\right)$, $\int_{t_0}^{t_1} \frac{dt}{t} = \ln\left(\frac{t_1}{t_0}\right)$ it is important that a shower gets also the soft emissions correct, in particular those terms which are both collinear and soft enhanced.

The soft emissions factorize again on the amplitude level, e.g.



$$= \frac{ig\varepsilon^\mu}{2p\cdot k}\, i\bar{u}(p)\gamma_\mu(\not p + \not k)\dots\dots$$

$$= -g\frac{\cancel{2}\,\varepsilon\cdot p}{\cancel{2}\,p\cdot k}\,(t^a)_{ij}\,(\mathcal{M}_n)_{\dots j\dots}$$

On the level of the cross section, this leads to

$$d\sigma_{n+1} = \frac{d\omega}{\omega}\,\frac{d\Omega}{2\pi}\,\frac{\alpha_s}{2\pi}\sum_{i,j}^{n} C_{i,j}\,W_{i,j}\,d\sigma_n$$

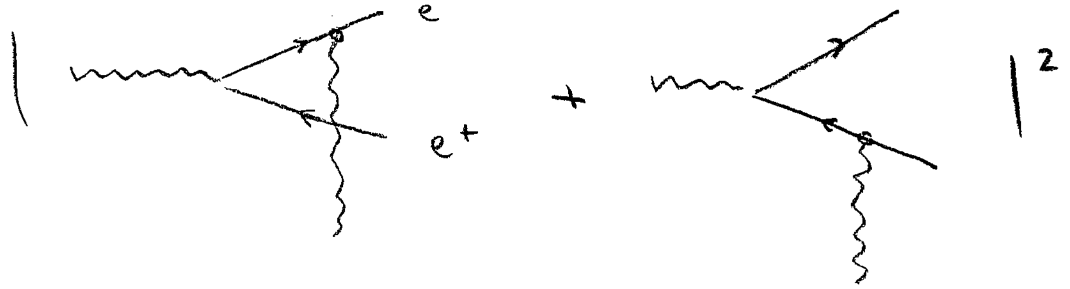
The sum is over external legs. C_{ij} is a color factor and

$$W_{i,j} = \frac{\omega^2\, p_i\cdot p_j}{p_i\cdot k\;\, p_j\cdot k} = \frac{1-\cos\Theta_{ij}}{(1-\cos\Theta_{ik})(1-\cos\Theta_{jk})}$$

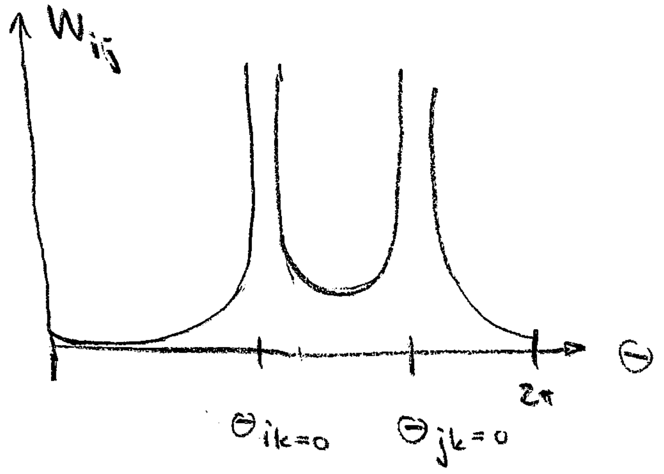
W_{ij} describes the interference of an emission from leg i with leg j.

This is a problem for the parton shower approach: we generate independent emissions from all legs, interference effects are missing.

In QED



At large angle the radiation interferes destructively. e^+e^- pair appears as an object of charge 0. Only at small angles individual charges are resolved.



let us split W_{ij} into two parts

$$W_{ij} = W_{ij}^{[i]} + W_{ij}^{[j]}$$

where

$$W_{ij}^{[i]} = \frac{1}{2}\left(W_{ij} + \frac{1}{1-\cos\Theta_{ik}} - \frac{1}{1-\cos\Theta_{jk}} \right)$$

$$W_{ij}^{[j]} = \frac{1}{2}\left(\qquad - \qquad + \qquad \right)$$

The function $W_{ij}^{[i]}$ has a remarkable property. If we write the angular integration as

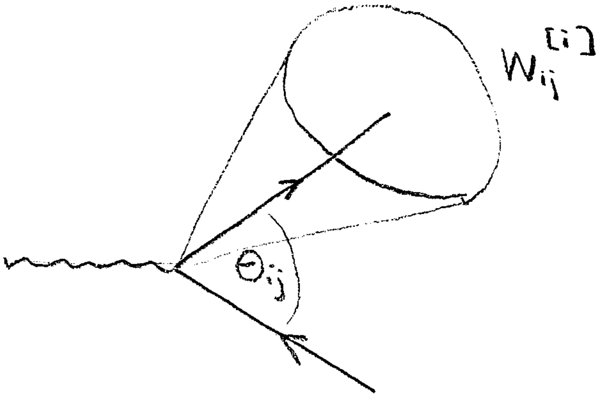
$$d\Omega = d\cos\Theta_{ik}\, d\varphi_{ik}$$

then one can show that

$$\int_0^{2\pi} \frac{d\phi_{ik}}{2\pi} W_{ij}^{[i]} = \frac{1}{1-\cos\Theta_{ik}} \quad \text{if } \Theta_{ik} < \Theta_{ij}$$

$$= 0 \quad \text{otherwise}$$

Averaged over the azimuthal angle, the soft radiation $W_{ij}^{[i]}$ has the same form as an emission from a single leg, except that there is no radiation outside an opening angle $\Theta_{iq} > \Theta_{ij}$.



The property that the azimuthally averaged radiation lies inside a cone is also called "angular ordering". For a proof that $W_{ij}^{[i]}$ has this property, see Ellis, Stirling & Webber.

This result implies that the parton shower gives the correct result if we use angular ordering: Instead of $t = p^2$, we should use the angle, or $1 - \cos\Theta$ as our ordering variable.

In this case, we obtain the correct pattern of soft emissions for quantities which are "azimuthally symmetric", i.e. which do not depend on <u>any</u> of the azimuthal angles of emissions.

There is one final complication: the soft emissions have color-structure $C_{ij} = -\sum_a T_i^a T_j^a \equiv -T_i \cdot T_j$ where T_i and T_j are the color generators of particles i and j. Note that color-charge conservation implies $\sum_i T_i = 0$. Using this property $\sum_{j \neq i} T_i \cdot T_j = -T_i^2 = -C_i \quad (C_F \text{ or } C_A) \quad (*)$

The parton shower uses C_i for the emissions from leg i. The soft emissions have structure

$\sum_{\substack{i,j \\ i \neq j}}^{n} -T_i \cdot T_j W_{ij}$ so there are non-trivial color correlations. The replacement $T_i \cdot T_j \to C_i$ is only correct in these cases

a.) for $n = 2$, since $T_1 = -T_2$; $T_1 \cdot T_2 = -T_1^2 = -T_2^2$

b.) for the leading soft + collinear log's, after azymuthal averaging.

c.) in the large-N_c limit

For b.) note that after the average $\int d\phi_{ik} W_{ik}^{[i]} = f(\cos\theta_{ki})$, independent of j, so we can use $(*)$.

To summerize: an angle ordered shower produces the correct

1.) collinear logarithms

2.) soft + collinear double log's for quantities which are azymuthally symetric

3.) but does not produce soft logarithms correctly

After the shower stops at an IR cut-off scale, generators like Pythia, Herwig and Sherpa use hadronisation models to turn the partons into hadrons. We have discussed the various approximations which go into the construction of these codes. However, in general these programs do quite an amazing job in simulating high-energy collider events and are used extensively in particular by experimentalists.