LO cross section for $e^+ e^- \to q \bar{q}$

```
In[•]:= << X`

Package-X v2.1.1, by Hiren H. Patel
For more information, see the guide

In[•]:= diag = ⟨v[p2, me], γμ, u[p1, me]⟩ ⊗
           ⟨u[p4, mq], γμ, v[p3, mq]⟩;

In[•]:= diagC = ⟨u[p1, me], γν, v[p2, me]⟩ ⊗
           ⟨v[p3, mq], γν, u[p4, mq]⟩;

In[•]:= diagC diag

Out[•]= ⟨v[p2, me], γμ, u[p1, me]⟩ ⊗ ⟨u[p4, mq], γμ, v[p3, mq]⟩ ⊗
         ⟨u[p1, me], γν, v[p2, me]⟩ ⊗ ⟨v[p3, mq], γν, u[p4, mq]⟩

In[•]:= ampSquared =
          e^4 eq^2 / Q^4
           (1/4 Spur[γμ, p1.γ + me 𝟙, γν, p2.γ - me 𝟙] ×
              Spur[γμ, p3.γ - mq 𝟙, γν, p4.γ + mq 𝟙] //
             Contract) // Simplify
```

Out[•]=
$$\frac{1}{Q^4}\, 4\, e^4\, eq^2 \left(d\, me^2\, mq^2 + 2\, p1.p4\, p2.p3 + 2\, p1.p3\, p2.p4 - 2\, me^2\, p3.p4 + d\, me^2\, p3.p4 + p1.p2\left((-2+d)\, mq^2 + (-4+d)\, p3.p4\right)\right)$$

```
In[•]:= kin = {p1.p2 → Q^2/2, p1.p3 → En^2 (1 - β Cos[θ]),
           p2.p4 → p1.p3, p1.p4 → En^2 (1 + β Cos[θ]),
           p2.p3 → p1.p4, p3.p4 → p1.p2, En → Q/2};
```

We neglect the electron mass

```
In[•]:= ampSquared2 = ampSquared /. me → 0 //. kin /. Q → 2 En //
          Simplify
```

Out[•]=
$$\frac{e^4\, eq^2\, \left(2\, (-3+d)\, En^2 + (-2+d)\, mq^2 + 2\, En^2\, \beta^2\, Cos[\theta]^2\right)}{2\, En^2}$$

In 4 dimensions...

```
In[•]:= ampSquared2 /. d → 4
```

Out[•]=
$$\frac{e^4\, eq^2\, \left(2\, En^2 + 2\, mq^2 + 2\, En^2\, \beta^2\, Cos[\theta]^2\right)}{2\, En^2}$$

```
In[•]:= integ =
         Integrate[
          (2 π) (α^2 eq^2)/(4 s) β (ampSquared2 /. d → 4) / (e^4 eq^2) /.
             Cos[θ] → Cθ // Expand, {Cθ, -1, 1}]
```

Out[•]=
$$\frac{eq^2\, \pi\, \alpha^2\, \beta^3}{3\, s} + 2\left(\frac{eq^2\, \pi\, \alpha^2\, \beta}{2\, s} + \frac{eq^2\, mq^2\, \pi\, \alpha^2\, \beta}{2\, En^2\, s}\right)$$

```
In[•]:= integ2 = 1/β integ /. β → Sqrt[1 - mq^2/En^2] // Expand
```

Out[•]=
$$\frac{4\, eq^2\, \pi\, \alpha^2}{3\, s} + \frac{2\, eq^2\, mq^2\, \pi\, \alpha^2}{3\, En^2\, s}$$

```
In[•]:= σtot = (4 eq^2 π α^2)/(3 s) (integ2 / (4 eq^2 π α^2)/(3 s) // Expand)
```

Out[•]=
$$\frac{4\, eq^2\, \left(1 + \frac{mq^2}{2\, En^2}\right)\, \pi\, \alpha^2}{3\, s}$$

In[•]:=
```
charges = {e[1] → 2 / 3, e[2] → -1 / 3, e[4] → 2 / 3,
   e[3] → -1 / 3, e[5] → 1 / 3, e[6] → 2 / 3};
```

In[•]:=
```
Sum[3 e[q]^2, {q, 1, 3}] /. charges
```

Out[•]= 2

Now do the same in *d* dimensions

Here, we also neglect the quark mass for simplicity

In[•]:=
```
massless = {mq → 0, k → En, β → 1, En → Q / 2};
```

To make the coupling dimensionless, we replace *e*— > $\mu^{\epsilon}\, e$

In[•]:=
```
ampSquared2 =
  μ^(4 ϵ) ampSquared /. me → 0 //. kin /. massless // Simplify
```

Out[•]= $e^4\, eq^2\, \mu^{4\epsilon}\left(-3 + d + \mathrm{Cos}[\theta]^2\right)$

In[•]:=
$$\mathrm{solidAngle}[d\_] = \frac{2\,\pi^{d/2}}{\mathrm{Gamma}[d/2]};$$

In[•]:=
$$\mathrm{phaseSpaced} = \frac{1}{2\,s}\left(\frac{1}{2\,(2\,\pi)^{d-1}}\right)^2 \frac{k^{d-2}}{En^2}\,\frac{En}{2\,k}\,(2\,\pi)^{d}$$

Out[•]=
$$\frac{2^{-2-d}\,k^{-3+d}\,\pi^{2-d}}{En\ s}$$

In[•]:=
```
angInt = Integrate[solidAngle[d - 2] Sin[θ]^(d-3),
   {θ, 0, π}, GenerateConditions → False]
```

Out[•]=
$$\frac{2\,\pi^{\frac{1}{2}(-1+d)}}{\mathrm{Gamma}\left[\frac{1}{2}(-1+d)\right]}$$

Check:

```
In[•]:= angInt / solidAngle[d - 1] // Simplify
```

Out[•]= 1

```
In[•]:= phaseSpacedFull =
         phaseSpaced solidAngle[d - 2] Sin[θ]^(d-3)
```

Out[•]=
$$\frac{2^{-1-d}\, k^{-3+d}\, \pi^{2+\frac{1}{2}(-2+d)-d}\, \mathrm{Sin}[\theta]^{-3+d}}{\mathrm{En\ s\ Gamma}\left[\frac{1}{2}(-2+d)\right]}$$

```
In[•]:= dSigmadCθ =
         1/Sin[θ] phaseSpacedFull ampSquared2 //. massless /.
            s → Q^2 /. Cos[θ] → Cθ /. Sin[θ]^a_ :→ (1 - Cθ^2)^(a/2) //
          Simplify
```

Out[•]=
$$\frac{2^{3-2d}\left(1-\mathrm{C}\theta^{2}\right)^{\frac{1}{2}(-4+d)}\left(-3+\mathrm{C}\theta^{2}+d\right) e^{4}\, \mathrm{eq}^{2}\, \pi^{1-\frac{d}{2}}\, Q^{-6+d}\, \mu^{4\epsilon}}{\mathrm{Gamma}\left[-1+\frac{d}{2}\right]}$$

```
In[•]:= sigTot = Integrate[dSigmadCθ /. d → 4 - 2 ϵ,
          {Cθ, -1, 1}, GenerateConditions → False]
```

Out[•]=
$$\frac{e^{4}(-1+\epsilon)\, \mathrm{eq}^{2}\, (4\pi)^{-1+\epsilon}\, Q^{-2(1+\epsilon)}\, \mu^{4\epsilon}\, \mathrm{Gamma}[2-\epsilon]}{(-3+2\epsilon)\, \mathrm{Gamma}[2-2\epsilon]}$$

```
In[•]:= sigTot2 = sigTot /. e → √(4 π α) /. Q → √Q2 //
          FullSimplify
```

Out[•]=
$$\frac{(-1+\epsilon)\, \mathrm{eq}^{2}\, (4\pi)^{1+\epsilon}\, \mathrm{Q2}^{-1-\epsilon}\, \alpha^{2}\, \mu^{4\epsilon}\, \mathrm{Gamma}[2-\epsilon]}{(-3+2\epsilon)\, \mathrm{Gamma}[2-2\epsilon]}$$

Test:

```
In[•]:= sigTot2 /. ϵ → 0
```

Out[•]=
$$\frac{4\, \mathrm{eq}^{2}\, \pi\, \alpha^{2}}{3\, \mathrm{Q2}}$$

In[•]:=
```
σtot /. mq → 0
```

Out[•]=
$$\frac{4\,\mathrm{eq}^2\,\pi\,\alpha^2}{3\,s}$$

2-particle phase space

In[•]:=
$$\text{solidAngle}[d\_] = \frac{2\,\pi^{d/2}}{\text{Gamma}[d/2]};$$

In[•]:=
```
solidAngle[4 - 1]
```

Out[•]= $4\,\pi$

In[•]:=
$$\text{phaseSpaced} = \left(\frac{1}{2\,(2\,\pi)^{d-1}}\right)^2 \frac{k^{d-2}}{\mathrm{En}^2}\,\frac{\mathrm{En}}{2\,k}\,(2\,\pi)^d\,\text{solidAngle}[d-1]\;/.\; k\to \mathrm{En}\;/.\;\mathrm{En}\to Q/2\;/.\;s\to Q^2\;/.\;d\to d\;/.\;Q\to\sqrt{Q2}\;/.\;d\to 4-2\,\epsilon\;//\;\text{FullSimplify}$$

Out[•]=
$$\frac{16^{-1+\epsilon}\,\pi^{-\frac{1}{2}+\epsilon}\,\mathrm{Q2}^{-\epsilon}}{\text{Gamma}\left[\frac{3}{2}-\epsilon\right]}$$

In[•]:=
$$\text{phaseSpaced}\;/.\;\mathrm{Q2}\to(2\,\text{Qhalf})^2\;/.\;\text{Solve}[d == 4-2\,\epsilon,\,\epsilon][[1]]\;//\;\text{Simplify}\;//\;\text{PowerExpand}$$

Out[•]=
$$\frac{2^{-d}\,\pi^{\frac{3}{2}-\frac{d}{2}}\,\text{Qhalf}^{-4+d}}{\text{Gamma}\left[\frac{1}{2}\,(-1+d)\right]}$$

In[•]:=
$$\frac{\text{solidAngle}[d-1]}{8\,(2\,\pi)^{d-2}}\;//\;\text{Simplify}$$

Out[•]=
$$\frac{2^{-d}\,\pi^{\frac{3}{2}-\frac{d}{2}}}{\text{Gamma}\left[\frac{1}{2}\,(-1+d)\right]}$$

In[∘]:= ratio = sigTot2 / phaseSpaced // FullSimplify

Out[∘]= $-\frac{64\ (-1+\epsilon)^2\ \text{eq}^2\ \pi^2\ \alpha^2\ \mu^{4\epsilon}}{-3+2\epsilon}$

In[∘]:= $\frac{2^{-d}\ \pi^{\frac{3}{2}-\frac{d}{2}}\ \text{Qhalf}^{-4+d}}{\text{Gamma}\left[\frac{1}{2}\ (-1+d)\right]}$ /. d → 4

Out[∘]= $\frac{1}{8\ \pi}$