1.) $G = SU_L(4) \times SU_R(4)$; $H = SU_V(4)$

$$\rightsquigarrow n_G = d(G) - d(H) = n_f^2 - 1 = 15.$$

2.) Q_A^a is time-independent, choose $Q_A^a = Q_A^a(t)$

if $y^\mu = (t, \vec{y})$.

Then we need $x^\mu = (t, \vec{x})$

$$\int d^3x \, \langle 0 | \Big[\underbrace{\bar{\psi}(x) \gamma^0 \gamma^5}_{a} t^a \underbrace{\psi(x)}_{b}, \underbrace{\bar{\psi}(y) \gamma_5}_{c} t^a \underbrace{\psi(y)}_{d} \Big] | 0 \rangle$$

$$= \int d^3x \, \langle 0 | \; \bar{\psi}(x) \gamma^0 \gamma^5 t^a \{ \psi(x), \bar{\psi}(y) \} \gamma^5 t^a \psi(y)$$

$$- \bar{\psi}(y) \gamma_5 t^a \{ \bar{\psi}(x) \gamma^0 \gamma^5 t^a, \psi(y) \} \psi(x) | 0 \rangle$$

where we used that $\{ \psi(x), \psi(y) \} = 0$ and $\{ \bar{\psi}(x), \bar{\psi}(y) \} = 0$ at equal times. The two nonvanishing commutators are obtained from

$$\{ \psi_{\alpha,i}(t, \vec{x}), \psi^\dagger_{\beta,j}(t, \vec{y}) \} = \overset{\text{Dirac}}{\delta_{\alpha\beta}} \, \overset{\text{quark flavor}}{\delta_{ij}} \, \delta^{(3)}(\vec{x} - \vec{y})$$

$$= \langle 0 | \bar{\psi}(y) \gamma^0 \gamma^5 t^a \gamma^0 \gamma^5 t^a \psi(y)$$

$$- \bar{\psi}(y) \gamma_5 t^a \gamma^0 \gamma^0 \gamma^5 t^a \psi(y) \; | 0 \rangle$$

$$= -2 \langle 0 | \bar{\psi}(y) (t^a)^2 \psi(y) | 0 \rangle$$

Because of $SU(n_f)$ invariance we can replace

$$(t^a)^2 = \frac{1}{n_f^2 - 1} \underbrace{\sum_a (t^a)^2}_{C_F} = \frac{1}{n_f^2-1} \frac{n_f^2 - 1}{2 n_f} \cdot \mathbb{1}$$

$$= - \frac{1}{n_f} \langle 0 | \bar{\psi}(y) \psi(y) | 0 \rangle$$

Due to translation invariance, we can set $y=0$.

Due to $SU(n_f)$ invariance each flavor gives the same result

$$= - \langle 0 | \bar{u}(0) u(0) | 0 \rangle = - \langle 0 | \bar{d}(0) d(0) | 0 \rangle$$

$$= \ldots$$

3.)

$$\mathcal{L} = \frac{F^2}{4} \langle \partial_\mu U^+ \partial^\mu U \rangle$$

$\swarrow$ set $F = 1$ for the moment

$$U(x) = \exp(i\, \vec{\sigma}\vec{\pi})$$

$$= \underbrace{\cos(|\vec{\pi}|)}_{} + i\, \vec{\pi}\cdot\vec{\sigma} \cdot \underbrace{\frac{1}{|\vec{\pi}|}\sin(|\vec{\pi}|)}_{(1 - \vec{\pi}^2/6 + \cdots)}$$

$$= 1 + i\vec{\pi}\vec{\sigma} - \frac{\vec{\pi}^2}{2} - i\vec{\pi}\cdot\sigma \cdot \vec{\pi}^2/6$$

$$\partial_\mu U = i\, \partial_\mu\vec{\pi}\cdot\vec{\sigma} - \vec{\pi}\partial_\mu\vec{\pi} - i\, \partial_\mu\vec{\pi}\cdot\vec{\sigma}\, \vec{\pi}^2/6$$

$$- i\,\vec{\pi}\cdot\vec{\sigma}\, \frac{1}{3}\vec{\pi}\partial_\mu\vec{\pi}$$

$$\langle \partial_\mu U \partial^\mu U^+ \rangle = \langle \partial_\mu\vec{\pi}\cdot\vec{\sigma}\, \partial^\mu\vec{\pi}\cdot\vec{\sigma} + (\vec{\pi}\partial_\mu\vec{\pi})^2$$

$$- 2\, \partial_\mu\vec{\pi}\cdot\vec{\sigma}\, \partial^\mu\vec{\pi}\cdot\vec{\sigma}\, \cdot \vec{\pi}^2/6$$

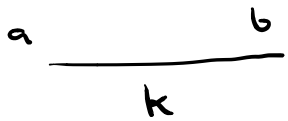
$$- 2\, \partial_\mu\vec{\pi}\cdot\vec{\sigma}\, \vec{\pi}\cdot\vec{\sigma}\, \frac{1}{3}\vec{\pi}\partial^\mu\vec{\pi} \rangle$$

Note: $\langle \sigma^a \sigma^b \rangle = \frac{1}{2} \langle \{ \sigma^a , \sigma^b \} \rangle = \langle \mathbb{1} \rangle \delta^{ab} = 2\delta^{ab}$

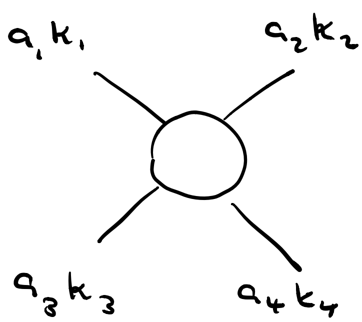
$$\rightsquigarrow \langle \partial_\mu U \partial^\mu U^\dagger \rangle = 2\, \partial_\mu \vec{\pi}\, \partial^\mu \vec{\pi} + 2 (\vec{\pi}\, \partial_\mu \vec{\pi})^2 - \frac{2}{3} \partial_\mu \vec{\pi}\, \partial^\mu \vec{\pi} \cdot \vec{\pi}^2 - \frac{4}{3} (\vec{\pi}\, \partial_\mu \vec{\pi})^2$$

$$\frac{F^2}{4} \langle \partial_\mu U \partial^\mu U^\dagger \rangle = \frac{1}{2} \partial_\mu \vec{\pi}\, \partial^\mu \vec{\pi} + \frac{1}{6F^2} (\vec{\pi}\, \partial_\mu \vec{\pi})^2 - \frac{1}{6F^2} \partial_\mu \vec{\pi}\, \partial^\mu \vec{\pi} \cdot \vec{\pi}^2$$

Feynman rules:



$$= \frac{i}{k^2} \delta^{ab}$$



$$\hat{=} \frac{i}{6F^2} \Big\{ - \delta^{a_1 a_2} \delta^{a_3 a_4} k_2 \cdot k_4 + \delta^{a_1 a_2} \delta^{a_3 a_4} k_1 \cdot k_2 + \text{"permutations"} \Big\}$$

$i\partial_\mu \hat{=} k$

$$\hat{=} \frac{i}{3F^2} \Big\{ \delta^{a_1 a_2} \delta^{a_3 a_4} \big(2 k_1 \cdot k_2 + 2 k_3 \cdot k_4 - k_1 \cdot k_3 - k_1 \cdot k_4 - k_2 \cdot k_3 - k_2 \cdot k_4 \big) + \text{“} 2 \leftrightarrow 3 \text{”} + \text{“} 2 \leftrightarrow 4 \text{”} \Big\}$$

5.)

To get the scattering amplitude, it is good enough to know the on-shell vertex and of course we can always use momentum conservation $\sum_i k_i = 0$.

With this, we have $s = (p_1 + p_2)^2$

$$\hat{=} \frac{i}{F^2} \Big\{ \delta^{a_1 a_2} \delta^{a_3 a_4}\, s + \text{“} 2 \leftrightarrow 3 \text{”} + \text{“} 2 \leftrightarrow 4 \text{”} \Big\}$$

$$+ \text{“off-shell} \propto p_i^2 \text{”}$$

$t = (p_1 + p_3)^2$ $\quad$ $u = (p_1 + p_4)^2$

$$\hat{=} i \,\big(\delta^{a_1 a_2} \delta^{a_3 a_4} s + \delta^{a_1 a_3} \delta^{a_2 a_4} t + \delta^{a_1 a_4} \delta^{a_2 a_3} u \big\}$$

$$= \frac{\quad}{F^2}\Big\{ \ \cdots \ s + \cdots \ u \Big)$$

$+$ off-shell

This last line __is__ the result for $\pi\pi$ scattering! We just need to replace $p_i \to -p_i$ for the outgoing momenta, i.e. $t = (p_1 - p_3)^2$ and $u = (p_1 - p_4)^2$.

6.) The external current part of $\mathcal{L}$ is

$$\Delta \mathcal{L} = l_\mu^i \, \bar{\Psi}_L \gamma^\mu t^i \Psi_L + r_\mu^i \, \bar{\Psi}_R \gamma^\mu t^i \Psi_R$$

u(2) matrix!

$$= \bar{\Psi}_L \gamma^\mu l_\mu \Psi_L + \bar{\Psi}_R \gamma^\mu r_\mu \Psi_R$$

Perform local chiral transformation $\Psi_L \to V_L(x) \Psi_L$

$$\mathcal{L}_{kin} = \bar{\Psi}_L i \not{\partial} \Psi_L \to \bar{\Psi}_L V_L^\dagger i \not{\partial} V_L \Psi_L$$

$$= \bar{\Psi}_L i \not{\partial} \Psi_L + \underbrace{\bar{\Psi}_L (V_L^\dagger i \not{\partial} V_L) \Psi}$$

$V_L(x)$

The current $\Delta \mathcal{L}$ yields

cancel!

$$\Delta \mathcal{L} \to \Delta \mathcal{L} - i \bar{\Psi}_L \underbrace{V_L^\dagger \gamma^\mu} (\partial_\mu V_L) \Psi_L$$