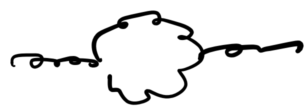
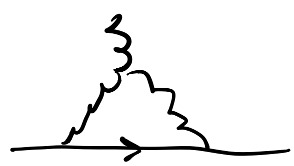
Exercise 4:

1.) a.) $L = I - V + 1 \quad (*)$



: $L = 1$, $I = 2$, $V = 2$ | $1 = 2 - 2 + 1$ ✓



: $L = 1$, $I = 3$, $V = 3$ | ✓

b.) $D = 4L - 2I$

c.) $4V = N + 2I$ & $(*)$

yield

$$2I = 4L + N - 4$$

$$\Rightarrow D = 4 - N < 0 \quad \text{for } N > 4$$

2.) The relations for the different particles read:

$$2V_{qg} = N_q + 2I_q$$

$$2V_{\eta g} = N_\eta + 2I_\eta$$

$$4V_{4g} + 3V_{3g} + V_{qg} + V_{\eta g} = N_g + 2I_g$$

The Euler relation is

$$L = 1 + I_q + I_\eta + I_g - V_{3g} - V_{4g} - V_{qg} - V_{\eta g}$$

Insert this into

$$D = 4L - 2I_g - I_q - 2I_\eta + V_{3g} + V_{4g}$$

$$\rightsquigarrow D = 4 - N_g - \frac{3N_q}{2} - N_\eta$$

However: the derivative in $V_{\eta g}$ acts on the outgoing ghost. For external outgoing ghosts, this never involves the loop momentum. We can thus subtract $N_\eta/2$ derivatives and get the stronger bound

$$D \leq 4 - N_g - \frac{3}{2}(N_q + N_\eta)$$

3.) $$\frac{d}{d\ln\mu} m_0 = 0 = \frac{d}{d\ln\mu} m Z_m$$

$$\longrightarrow \quad Z_m \underbrace{\frac{d}{d\ln\mu} m}_{\gamma_m \cdot m} + m \frac{d}{d\ln\mu} Z_m = 0$$

$$\longrightarrow \quad \gamma_m = - Z_m^{-1} \frac{dZ_m}{d\ln\mu}$$

$$\rightsquigarrow \quad Z_m \gamma_m = - \frac{dZ_m}{d\alpha_s} \underbrace{\frac{d\alpha_s}{d\ln\mu}}_{\beta(\alpha_s, \varepsilon)}$$

$$\longrightarrow \quad Z_m \gamma_m(\alpha_s, \varepsilon) = - \frac{dZ}{d\alpha_s} \left(\beta(\alpha_s) - 2\varepsilon\alpha_s \right)$$

Take the limit $\varepsilon \to \infty$, expand

$$\gamma_m(\alpha_s, \varepsilon) = \gamma_m(\alpha_s) + \varepsilon \gamma_m^{[1]} + \varepsilon^2 \gamma_m^{[2]}$$

$$+ \cdots$$

Leading term on RHS is

$$+ 2\alpha_s \frac{dZ_m^{[-1]}}{d\alpha_s} + O\left(\frac{1}{\varepsilon}\right)$$

$$\rightsquigarrow \gamma_m^{[i]} = 0 \quad i > 0$$

$$\gamma_m = 2\alpha_s \frac{dZ_m^{[-1]}}{d\alpha_s}$$