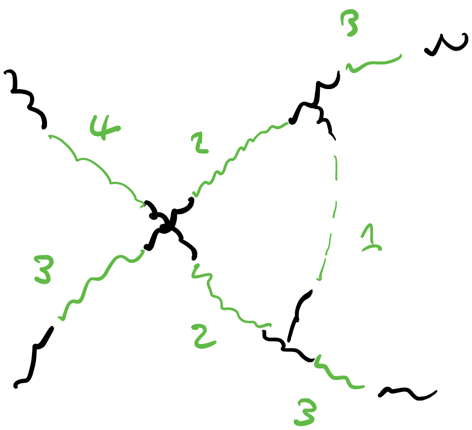
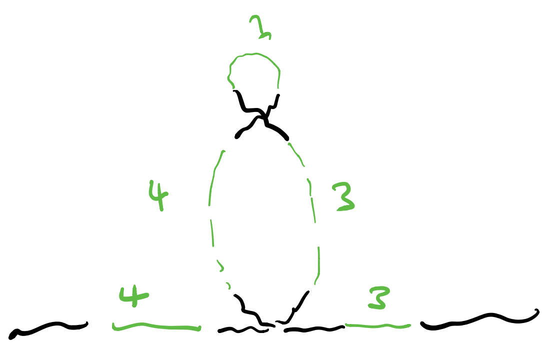
1.)



$$S = \frac{4!\;2\cdot 3\cdot 3}{4!\;3!\;3!} = 1/2$$



$$S = \frac{4\cdot 3\;\;4\cdot 3}{(4!)^2} = \frac{1}{4}$$

2.)

Infinitesimal form:

$$U_R(\alpha)\, t_R^a\, U_R^{-1}(\alpha) = (1 + i\alpha^b t_R^b)\, t_R^a \cdot (1 - i\alpha^c t_R^c) + O(\alpha^2)$$

$$= t_R^a + i\alpha^b (t_R^b t_R^a - t_R^a t_R^b) + O(\alpha^2)$$

$$= t_R^a + i\alpha^b \left(i f_{bac} \right) t_R^c + O(\alpha^2)$$

$$= \left(\delta_{ac} - i\alpha^b (t_A^b)_{ac} \right) t_R^c + O(\alpha^2)$$

note the minus sign!

$$= \mathcal{U}_A^{ac}(-\alpha)\, t_R^c + O(\alpha^2)$$

To prove it for all α one can split finite transformation into n small ones, take limit $n \to \infty$, i.e.

$$U(\alpha) = \lim_{n \to \infty} \left(1 + i \frac{\alpha^a}{n} t^a \right)^n$$

Check that the minus sign is correct, i.e. that we obtain the $[U_A(\alpha)]^{-1}$: Under a global $SU(N_c)$ transformation $\bar{\psi} \not{A}^a t^a \psi$ must be invariant.

we have $\psi \to U(\alpha)\psi$

$$\bar{\psi}^+ \to \bar{\psi}\, U^+(\alpha) = \bar{\psi}\, U^{-1}(\alpha)$$

$$A_\mu^a \to \mathcal{U}_A^{ab} A_\mu^b$$

$$\bar{\psi} \; \mathcal{U}^{-1}(\alpha)\, t^a\, \mathcal{U}(\alpha)\, \mathcal{U}_A^{ab}(\alpha)\, A^b\, \psi$$

$$= \bar{\psi} A^b t^b \psi \qquad \text{for arbitrary } \psi, \bar{\psi}, A$$

$$\leadsto \quad \mathcal{U}^{-1}(\alpha)\, t^a\, \mathcal{U}(\alpha)\, \mathcal{U}_A^{ab} = t^b$$

$$\leadsto \quad \mathcal{U}^{-1}(\alpha)\, t^c\, \mathcal{U}(\alpha) = t^b \left[\mathcal{U}_A^{-1}(\alpha)\right]^{bc}$$

$$= \left[\mathcal{U}(\alpha)\right]^{cb} t^b$$

structure constants are anti-symm.

The relation derived in the exercise is the ~~same~~

but with $\alpha \to -\alpha$. ✓

4.)

$$[C_{R',R}(n),\ t_R^b]$$

$$= M_{R'}^{a_1 \dots a_n} [t_R^{a_1} \dots t_R^{a_n},\ t_R^b]$$

$$= M_{R'}^{a_1 \dots a_n} \Big(t_R^{a_1} \dots t_R^{a_{n-1}} \overbrace{[t_R^{a_n},\ t^b]}^{i f^{a_n b c_n} t^{c_n}} + \underbrace{[t_R^{a_1} \dots t_R^{a_{n-1}},\ t^b]}_{= \dots} t^{a_n} \Big)$$

$$= -i M_{R'}^{a_1 \dots a_n} f^{b a_n c_n} t_R^{a_1} \dots t_R^{a_{n-1}} t_R^{c_n}$$

$$+ \dots$$

Rename: $a_n \to \hat{a}_n$; $c_n \to a_n$

$$= -i \sum_i f^{b \hat{a}_i a_i} M^{a_1 \dots \hat{a}_i \dots a_n} t_R^{a_1} \dots t_R^{a_n}$$

$\underbrace{\hspace{4cm}}_{= 0, \text{ see } 3.)}$

$= 0$ ✓