Exercise 1: (Partial) Solutions

3.)

$$f^{abc} = -2i\, \mathrm{tr}\left[t^a t^b t^c - t^b t^a t^c\right]$$

$$f^{abc} - (f^{abc})^*$$

$$= -2i\, \mathrm{tr}\left[t^a t^b t^c - t^b t^a t^c + t^c t^b t^a - t^c t^a t^b \right]$$

$t^\dagger_c = t_c$, etc.

$$= 0 \quad \checkmark$$

7.) Consider:

$$t^a_{ij}\, t^a_{ke} + \frac{1}{2N}\,\delta_{ij}\,\delta_{ke} =$$

Any hermitian matrix can be written as

$$M = c_0\,\mathbb{1} + c_a\, t^a$$

Note: $\mathrm{Tr}\left[t^b M\right] = c_a\, \mathrm{tr}\left(t^b t^a\right)$

$$= T_F\, c^b = \frac{1}{2}\, c^b$$

$$\mathrm{tr}\left[\mathbb{1} M\right] = c_0\, \mathrm{tr}\left(\mathbb{1}\right) = c_0 \cdot N$$

$$\rightarrow M = \frac{1}{N}\,\mathrm{tr}\left[M\,\mathbb{1}\right]\mathbb{1} + 2 \cdot \mathrm{tr}\left[M\, t^a\right] t^a$$

In components:

$$M_{ij} = \frac{1}{N} M_{ek} \delta_{ke} \delta_{ij}$$

$$+2 M_{lk} t^a_{ke} t^a_{ij} = M_{lk} \delta_{ie} \delta_{jk}$$

This must hold for any M_{ek}

$$\rightarrow \quad t^a_{ij} t^a_{ke} = -\frac{1}{2N} \delta_{ij} \delta_{ke} + \frac{1}{2} \delta_{ie} \delta_{jk}$$