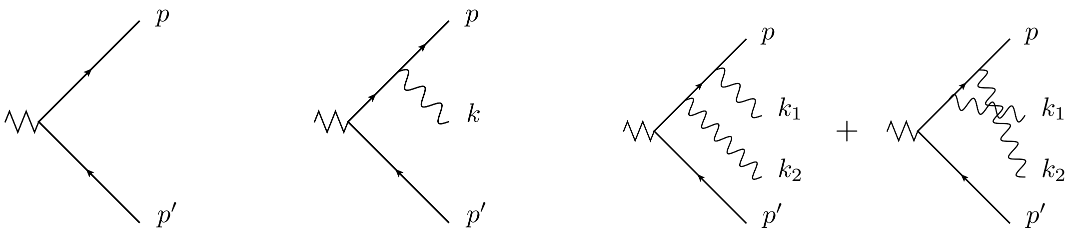
1. The following Feynman diagrams



describe the production of an $e^-(p)e^+(p')$ pair from a virtual photon (first diagram) together with one (second diagram) or two soft photons (last two diagrams) emitted from the outgoing electron. The result for the first diagram is $\mathcal{M}_0 = -ie\bar{u}(p)\gamma^\mu v(p')$. We consider the diagrams in the soft limit where the outgoing photons are soft, i.e. k, k_1, $k_2 \to 0$.

(a) Show that the leading term in the expansion around the soft limit is

$$\mathcal{M}_1(k) = \mathcal{S}_1(k)\mathcal{M}_0 = e\frac{\varepsilon^* \cdot p}{k \cdot p}\mathcal{M}_0\,, \tag{1}$$

for the emission of a single photon from the electron (second diagram), where ε is the polarization vector of the photon.

(b) Show that the result for the emission of two photons from the electron (i.e. the sum of the last two diagrams) is

$$\mathcal{M}_2(k_1,k_2) = \mathcal{S}_2(k_1,k_2)\mathcal{M}_0 = \frac{e^2}{2}\frac{\varepsilon_1^* \cdot p}{k_1 \cdot p}\frac{\varepsilon_2^* \cdot p}{k_2 \cdot p}\mathcal{M}_0 = \frac{1}{2!}\mathcal{S}_1(k_1)\mathcal{S}_1(k_2)\mathcal{M}_0\,. \tag{2}$$

2. Consider the following QED Wilson line operator

$$S_v(y) = \exp\left[-ie\int_0^\infty ds\, v \cdot A(y+vs)\, e^{-\delta s}\right]\,.$$

The factor $e^{-\delta s}$ was inserted to ensure that the integral converges for $s \to \infty$ and one takes $\delta \to 0$ after evaluating the matrix elements of the operator.

Compute the matrix elements of the operator $S_v(0)$ with final state photons

$$\mathcal{S}_1(k) = \langle k, \varepsilon | S_v(0) | 0 \rangle\,,$$
$$\mathcal{S}_2(k_1,k_2) = \langle k_1, \varepsilon_1;\, k_2, \varepsilon_2 | S_v(0) | 0 \rangle$$

and show that the results are identical to the one obtained in the previous exercise if $v^\mu \propto p^\mu$, i.e. if the reference vector v^μ is chosen along the direction of the electron momentum. The state $|k, \varepsilon\rangle$ contains of photon of polarization λ and associated polarization vector $\varepsilon(\vec{k}, \lambda)$.

3. Using the method of regions, verify the leading term in the expansion of the following integral

$$I = \int_0^\infty dk \, \frac{k^{1+\epsilon}}{(k^2+m^2)\,(k^2+M^2)} = -\frac{1}{M^2}\left[\ln\frac{m}{M} + \mathcal{O}\left(\frac{m^2}{M^2}\right)\right] + \mathcal{O}(\epsilon)\,.$$

To obtain the expansion, expand the integrand in the following two regions

hard:	$k \sim M\,,$
soft:	$k \sim m\,,$

compute the contributions, add up and take the limit $\epsilon \to 0$ at the end. The following integral is useful

$$\int_0^\infty dx \, \frac{x^a}{(1+x)^b} = \frac{\Gamma(a+1)\Gamma(b-a-1)}{\Gamma(b)}\,.$$

4. Consider two light-cone vectors n_μ and $\bar{n}_\mu$, with $\bar{n} \cdot n = 2$. Show that the operators

$$P_+ = \frac{\not{n}\not{\bar{n}}}{4}\,, \qquad\qquad P_- = \frac{\not{\bar{n}}\not{n}}{4}\,,$$

are projection operators with $P_+ + P_- = 1$.

5. Use the projection operators $P_\pm$ to split the quark field into two components

$$\psi(x) = \xi(x) + \eta(x) = P_+\psi(x) + P_-\psi(x)\,.$$

Show that

(a) $\not{n}\,\xi(x) = 0$,
(b) $\bar{\xi}(x)\,\xi(x) = 0$,
(c) $\bar{\xi}(x)\,\not{D}_\perp\xi(x) = 0$,
(d) $\bar{\xi}(x)\gamma^\mu\xi(x) = n^\mu\,\bar{\xi}(x)\frac{\not{\bar{n}}}{2}\xi(x)$.