1. The S-matrix is related to the transfer matrix T via

$$\boldsymbol{S} = \boldsymbol{1} + i\boldsymbol{T}\,.$$

   (a) Show that the unitarity of the S-matrix implies

$$-i\left(\boldsymbol{T} - \boldsymbol{T}^\dagger\right) = \boldsymbol{T}\boldsymbol{T}^\dagger \tag{1}$$

   for the T-matrix. The usual scattering amplitude $\mathcal{M}$ of two particles into a final state with momenta $k_1,\, k_2, \ldots k_n$ is obtained from the matrix element

$$\langle k_1,\, k_2, \ldots k_n | \,\boldsymbol{T}\, | p_1\, p_2 \rangle = \mathcal{M}(p_1,\, p_2 \to k_1, \ldots k_n)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - \sum_i k_i)$$

   (b) Consider the unitarity relation (1) and take the $2 \to 2$ matrix element. Show that, after inserting a full set of states on the right-hand side, this leads to

$$i\Big[\mathcal{M}^*(k_1, k_2 \to p_1,\, p_2) - \mathcal{M}(p_1,\, p_2 \to k_1,\, k_2)\Big] = \sum_X \mathcal{M}(p_1,\, p_2 \to k_X)\mathcal{M}^*(k_1,\, k_2 \to k_X)(2\pi)^4\delta^{(4)}(p_1 + p_2 - k_X)$$

   Note: The sum over all states on the right-hand state includes integrals over the phase-spaces since we have to sum over all possible kinematic configurations.

   (c) Then, considering forward scattering and rewriting the right-hand side as the total cross section, we obtain the usual form of the optical theorem

$$2\,\mathrm{Im}\,\mathcal{M}(p_1,\, p_2 \to p_1,\, p_2) = F\,\sigma_{\mathrm{tot}}$$

   where $F = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$ is the usual flux factor and σ_{tot} is the cross section for the scattering of the initial state with $p_1,\, p_2$ into an arbitrary final state.

2. For the total cross section of e^+e^- into hadrons, one is able to write the optical theorem in especially simple by evaluating the leptonic part of the cross section, after which it reads

$$\sigma(e^+e^- \to \text{hadrons}) = -\frac{4\pi\alpha}{s}\,\mathrm{Im}\,\Pi_h(s) \tag{2}$$

where the hadronic vacuum polarization $\Pi_h(s)$ is given by the vector current two-point function

$$\Pi_h^{\mu\nu}(q^2) = \int d^4x \langle 0 | \, T \left\{ J^\mu(x) \, J^\nu(0) \right\} \, |0\rangle \tag{3}$$

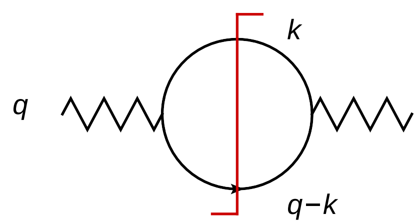
via

$$\Pi_h^{\mu\nu}(q^2) = \left(g^{\mu\nu} q^2 - q^\mu q^\nu \right) \Pi_h(q^2) \, . \tag{4}$$

The quark electromagnetic current is

$$J^\mu(x) = \sum_f e_f \, \bar{\psi}_f(x) \, \gamma^\mu \, \psi_f(x) \, .$$

The goal of this exercise is to verify (2) at leading order by computing the imaginary part of the one-loop contribution



(a) Write down the loop diagram. Neglect fermion masses and consider $g_{\mu\nu}\Pi_h^{\mu\nu}$ to be able to work with a scalar quantity. Simplify the numerator by evaluating the fermion trace.

(b) Rewrite the integration over the loop momentum k in the form

$$\int d^dk = \int d^dk \int d^dr \, \delta^{(d)}(q - k - r)$$

so that r is the momentum flowing through the lower fermion line.

(c) Use the Cutkosky rules, which state that each cut propagator is replaced by

$$\frac{i}{p^2 - m^2 + i\epsilon} \to \theta(p^0)(2\pi)\delta(p^2 - m^2) \, ,$$

to obtain the imaginary part. Show that this leads to the two-particle phase-space integral which we have evaluated earlier. Since the imaginary part is finite, it can be evaluated directly in $d = 4$.

(d) Take the result for $\mathrm{Im}\, \Pi_h(s)$, plug into (2) and verify that it agrees with our earlier result

$$\sigma(e^+e^- \to \text{hadrons}) = \frac{4\pi\alpha^2}{3s} \, N_c \sum_f e_f^2 \, .$$