QCD **Exercise 7**

Phase-space in dimensional regularization. IR safety. 19.11.20

1. Compute the massless two-particle phase space in $d = 4 - 2\epsilon$ dimensions.

2. An observable $\mathcal{O}$ is soft and collinear safe, if it does not change its value under soft and collinear emissions, i.e. if

   1.) $\lim_{k\to 0} \mathcal{O}_{n+1}(p_1, \ldots, p_n, k) = \mathcal{O}_n(p_1, \ldots, p_n)$,

   2.) $\mathcal{O}_{n+1}(p_1, \ldots p_n, p_{n+1}) = \mathcal{O}_n(p_1, p_n + p_{n+1})$ if $p_n \parallel p_{n+1}$.

   Decide which of the following observables are IR safe:

   (a) The number of particles produced in a collision.

   (b) The total energy deposited in a fixed angular region.

   (c) The energy of the hardest particle.

   (d) The invariant mass of all particles in a hemisphere (defined by a fixed reference vector $\vec{n}$).

3. Derive the distribution relation

$$x^{-1+\alpha} = \frac{1}{\alpha}\delta(x) + \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left[\frac{\ln^n(x)}{x}\right]_+$$

   valid for test functions $\phi(x)$ integrated over the interval $0 \leq x \leq 1$. The +-distributions are defined as

$$\int_0^1 [f(x)]_+ \, \phi(x) = \int_0^1 f(x) \left[\phi(x) - \phi(0)\right].$$

   To obtain the above identity write

$$\int_0^1 dx \, x^{-1+\alpha} \, \phi(x) = \int_0^1 dx \, x^{-1+\alpha} \, \{[\phi(x) - \phi(0)] + \phi(0)\}$$

   The first term inside the curly brackets gives rise to the + distributions, the second one the delta functions.

4. Use the identity of the previous exercise to compute the integrals

$$I_1 = \int_0^1 dx \, x^{-1+\epsilon} \, (1-x)^{2+\epsilon}$$

$$I_2 = 2\int_0^{\pi/2} d\theta \sin^{-1+\epsilon}(\theta) = \int_0^{\pi} d\theta \, |\sin(\theta)|^{-1+\epsilon}$$

   up to terms suppressed by ϵ.