QCD Exercise 4

Superficial degree of divergence, anomalous dimensions 22.10.20

1. Superficial degree of divergence. Euler has derived an identity which can be used to relate the number of loops, internal lines (i.e. propagators) and vertices in a connected diagram. It reads

$$L = I - V + 1\,. \tag{1}$$

   (a) Verify this identity for a self-energy and a vertex one-loop diagram in QCD.

   (b) By looking at the integrand in momentum space, show that in ϕ^4 theory, the superficial degree of divergence of an irreducible loop diagram is

$$D = 4L - 2I\,.$$

   (c) Use (1) to show that diagrams with more than $N = 4$ external legs have $D < 0$ in ϕ^4 theory and are thus superficially convergent. More specifically one can derive

$$D = 4 - N\,.$$

   To show this, use the fact that each ϕ^4 vertex has four legs, which are either external legs and internal legs connecting two vertices

$$4V = N + 2I\,. \tag{2}$$

   (d) *Bonus:* Prove Euler's formula (1).

2. In the lecture, we have derived the superficial degree of divergences of a QCD diagram as

$$D = 4L - 2I_g - I_q - 2I_\eta + V_{3g} + V_{\eta g}\,.$$

   The formula takes into account that the gluon and ghost propagators are quadratic, while the fermion propagator is linear. The three-gluon vertices V_{3g} and the ghost-gluon vertex $V_{\eta g}$ involve a derivative coupling. Following similar steps as in Exercise 1, derive the QCD result

$$D = 4 - \frac{3}{2}N_q - N_g - \frac{3}{2}N_\eta\,, \tag{3}$$

   where N_g, N_q and N_η are the number of external gluons, quarks and ghosts. The result (3) implies that also in QCD the superficial degree of freedom is negative for Green's functions with more than four external legs.

Hints: Derive first a relation analogous to (2) between the number of vertices, propagators and external legs for each of the QCD fields. Then, proceeding as in Exercise 1, one finds

$$D = 4 - \frac{3}{2}N_q - N_g - N_\eta \,, \tag{4}$$

The stronger bound (3) is found by using that the derivative in $V_{\eta g}$ acts on the outgoing ghost field. The derivatives on the external outgoing ghost fields do not contribute to D.

3. In the lecture we derived the "magic" relation

$$\beta(\alpha_s) = 4\alpha_s^2 \frac{dZ_g^{[-1]}}{d\alpha_s}$$

for the β-function using the fact that the fact that the bare coupling is μ independent. This relation holds in the Minimal Subtraction (MS) scheme, where the renormalization factors are pure pole terms

$$Z_i = 1 + \sum_{k=1}^{\infty} \frac{1}{\epsilon^k}\, Z_i^{[-k]}(\alpha_s)\,.$$

Use the same strategy to derive the relation

$$\gamma_m(\alpha_s) = 2\alpha_s \frac{dZ_m^{[-1]}}{d\alpha_s}\,,$$

where Z_m is the mass renormalization in the MS scheme and $\gamma_m(\alpha_s)$ the associated anomalous dimension

$$\frac{d}{d\ln\mu} m_f(\mu) = \gamma_m(\alpha_s)\, m_f(\mu)\,.$$