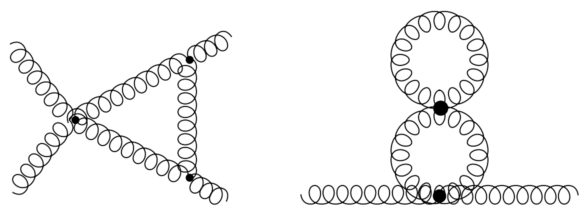
QCD Exercise 3

Casimir invariants, running coupling 15.10.20

1. What are the symmetry factors for the two diagrams below?



2. Derive the identity
$$U_R(\alpha)\, t_R^a\, U_R^{-1}(\alpha) = t_R^b\, U_A^{ba}(\alpha), \tag{1}$$
where
$$U_R(\alpha) = \exp(i\alpha^a t_R^a)$$
is a group element in the representation R. The quantity $U_A^{ab}(\alpha)$ is the same group element in the adjoint representation. *Hint:* Prove the infinitesimal form of (1).

3. Invariant tensors. The trace
$$M_R^{a_1 \dots a_n} = \mathrm{tr}(t_R^{a_1} \dots t_R^{a_n})$$
is invariant under the substitution $t_R^{a_i} \to U_R(\alpha)\, t_R^{a_i}\, U_R^{-1}(\alpha)$. Using (1), show that this implies
$$\sum_i f_{a\hat{a}_i a_i} M_R^{a_1 \dots \hat{a}_i \dots a_n} = 0 \tag{2}$$
in infinitesimal form, where $\hat{a}_i$ is inserted in position i.

4. Casimir Operators. Show that (2) implies that the Casimir operators
$$C_{R,R'}(n) = \mathrm{tr}(t_{R'}^{a_1} \dots t_{R'}^{a_n})\, t_R^{a_1} \dots t_R^{a_n}$$
commute with all generators. According to Schur's lemma this then implies that $C_{R,R'}(n)$ is a constant times the unit matrix if R is irreducible.

5. Approximate solution of the RG for the strong coupling. Define $a(L) = \alpha_s(\mu)/(4\pi)$ so that the RG equation reads

$$\frac{da}{dL} = \beta(a)/2 = -a(\beta_0 a + \beta_1 a^2 + \ldots)\,,$$

where $L = \ln(\mu^2/\Lambda^2)$. This equation can be solved using separation of variables,

$$L = \int \frac{da}{\beta(a)/2}\,,$$

but when including higher-order terms, one does not obtain an analytic expression for $a \equiv a(L)$. Solve the equation iteratively by expanding

$$\int \frac{da}{\beta(a)/2} = -\int \frac{da}{\beta_0 a^2}\left[1 - \frac{\beta_1 a}{\beta_0} + \ldots\right] = \frac{1}{a\beta_0} + \frac{\beta_1 \ln a}{\beta_0^2} + \ldots$$

and using the lower-order solution in the higher-order terms.

(a) Show that this leads to the expression

$$a = \frac{1}{\beta_0 L} - \frac{\beta_1 \ln L}{\beta_0^3 L^2} + \ldots\,. \tag{3}$$

Note: simply solving the equation leads to $\ln \beta_0 L$ instead of $\ln L$ on the right-hand side of the equation, but the $\ln \beta_0$ terms can be absorbed into a redefinition of Λ.

(b) What is the meaning of the scale Λ and what is its approximate numerical value, if we use the world average $\alpha_s(M_Z = 91.2\,\mathrm{GeV}) = 0.118$ as an input? To get the numerical value use

$$\begin{aligned}
\beta_0 &= \frac{11}{3}\,C_A - \frac{4}{3}\,T_F n_f\,, \\
\beta_1 &= \frac{34}{3}\,C_A^2 - \frac{20}{3}\,C_A T_F n_f - 4 C_F T_F n_f\,,
\end{aligned}$$

What is the appropriate value for n_f in (3)? The answer to this question is nontrivial and will be discussed in the lectures. Feel free to choose $n_f = 6$ for the determination of Λ.