QCD — Serie 1

Lie Algebra of SU(N) — 23.9.20

1. Show that the generators t^a for a unitary transformation
$$g(\alpha) = \exp(i\,\alpha^a t^a)$$
are hermitian and that the generators are traceless, if we impose $\det g(\alpha) = 1$.

2. Show that the structure constants of SU(N) can be obtained from the trace
$$if^{abc} = 2\,\mathrm{tr}\big([t^a, t^b]t^c\big)\ . \tag{1}$$
Remember that we have normalized the generators of the fundamental representation as
$$\mathrm{tr}\big(t^a t^b\big) = T_F\,\delta^{ab} \quad \text{with} \quad T_F = \frac{1}{2}\,. \tag{2}$$

3. Use hermeticity of the generators and the result (1) of the previous exercise to show that the structure constants f_{abc} are real.

4. Derive the color identities (repeated indices are summed over)
$$f^{abc}t^b t^c = \frac{i}{2} C_A t^a\,, \tag{3}$$
$$t^a\,t^b\,t^a = (C_F - C_A/2)\,t^b\,. \tag{4}$$
Here, C_F and C_A are the values of the quadratic Casimir invariants of the fundamental and the adjoint representation, i.e.
$$t^a t^a = C_F\,\mathbf{1}\,, \qquad\qquad if^{acd}\,if^{cbd} = C_A\,\delta^{ab}\,. \tag{5}$$

5. Derive the value of C_F by taking the trace in (5) using (2).

6. Proof the Fierz identity for generators of the fundamental representation of SU(N)
$$t^a_{ij}\,t^a_{kl} = \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right)\ . \tag{6}$$
To derive the identity, use the completeness relation, namely that any $N \times N$ matrix M can be written as a linear combination of the $N^2 - 1$ generators and the identity matrix:
$$M = c_0\mathbf{1} + c_a t^a\,.$$
The coefficients c_0 and c_a can be extracted taking traces.

7. Use the Fierz identity (6) to compute the quadratic Casimirs C_A. To do so, use the definition (5). *Hint:* Use $\mathrm{tr}\big([t^a, t^c][t^c, t^b]\big) = if^{acd}\,if^{cbd}\,T_F$.