Renormalization

As is common in QFT, we encounter ultraviolet (UV) divergences when we compute quantum corrections. In a renormalizable theory, these can be absorbed into renormalized parameters and the field normalizations.

To control the divergences, one needs to regularize the theory. This regularization can destroy some symmetries of the theory, e.g. a hard cutoff on momenta destroys Lorentz invariance and gauge invariance.

It can be very cumbersome to restore these symmetries by hand after renormalization.

There is only one known regularization which preserves both gauge & Lorentz invariance: dimensional regularization (dim.reg.). For this reason all modern perturbative QFT computations are performed using dim. reg.

In principle, renormalization in QED and QCD is similar, but the schemes which are used are different. In QED one typically chooses physical quantities, e.g. m_e and then absorbs the divergences into these. In QCD, the low-energy dynamics is complicated and there is no physical quark mass and no physical coupling analogous to $\alpha = \alpha(q^2=0)$ in QED.

Instead, one simply absorbs divergences into parameters in $\mathcal{L}$ to obtain finite Green's functions. Since the parameters are not physical, they typically depend on the renormalization scale μ.

Let us define renormalized parameters:

$$g_0 = Z_g \, g_s \, \mu^{\varepsilon}$$

bare

renormalized $g_s \equiv g_s(\mu)$

$$m_{f_0} = Z_m \, m_f$$

$$\xi_0 = Z_\xi \, \xi$$ ← gauge param.

We also define renormalized fields

$$A^a_{0\mu} = Z_A^{1/2} A^a_\mu \,,$$

$$\psi_0 = Z_\psi^{1/2} \psi \,,$$

$$\eta_0 = Z_\eta^{1/2} \eta \,,$$

We then expend

$\alpha_s = \frac{g_s^2}{4\pi}$

$$Z_i = 1 + \frac{\alpha_s}{4\pi}\left[\frac{c_i^{[-1]}}{\varepsilon} + c_i^{[0]} \right] + \frac{\alpha_s^2}{4\pi}\left[\frac{c_i^{[-2]}}{\varepsilon^2} + \frac{c_i^{[-1]}}{\varepsilon} + c_i^{[0]} \right]$$

where $d = 4 - 2\varepsilon$ is the space-time dimension.

Divergences show up as $\frac{1}{\varepsilon}$ poles in loop integrals.

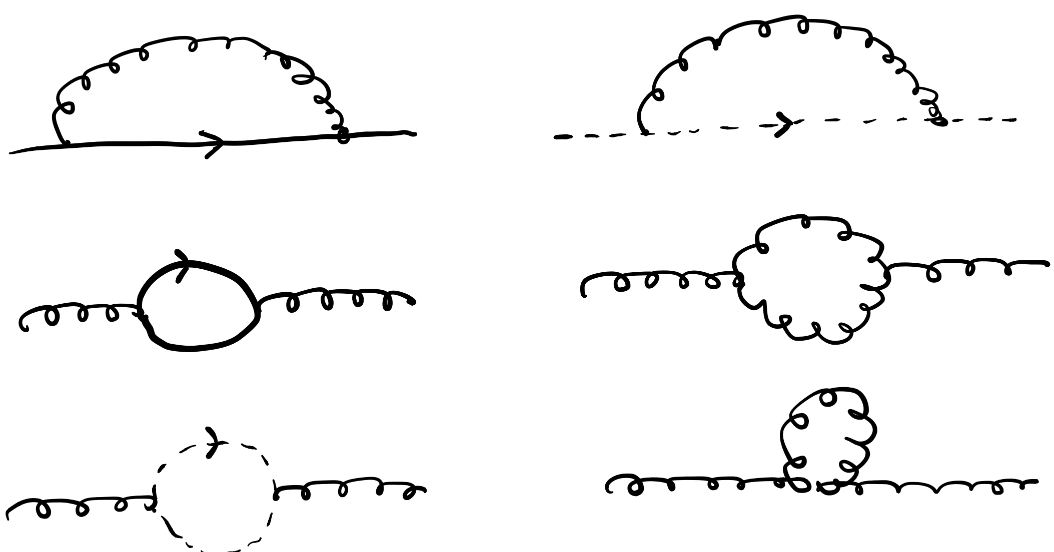
To renormalize at one loop, we must compute all one-particle irreducible (1PI) Green's functions with UV divergences. It is easy to see that only Green's functions with ≤ 4 external legs can be divergent:

<u>0-point & 1-point</u>

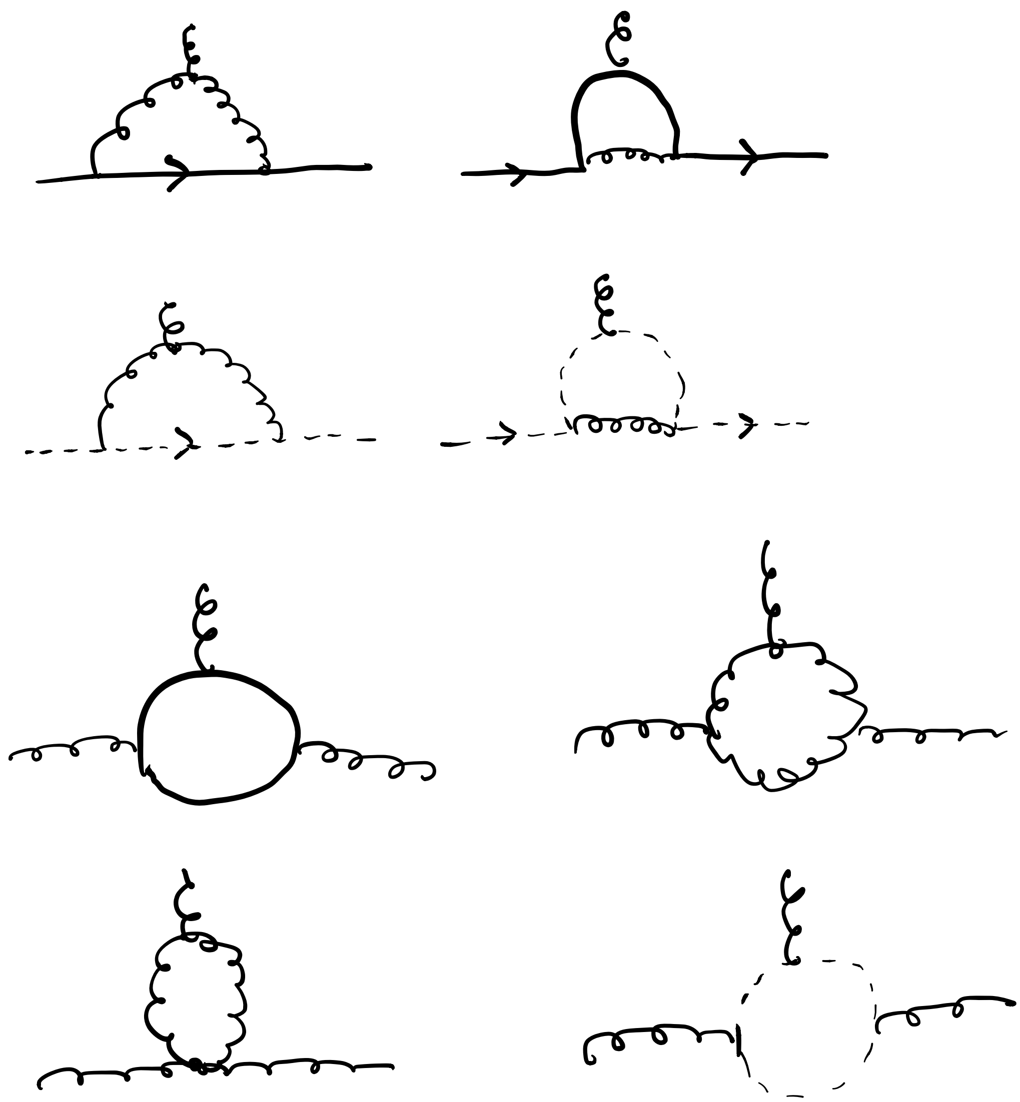
0-point only gives unobservable vacuum energy shift

1-point functions vanish in dim. reg.

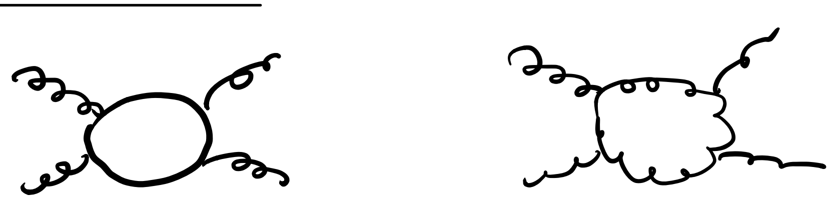
<u>2-point</u>



3-point



4-point



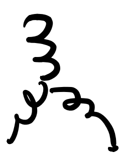
....

To investigate whether a diagram is UV divergent, one counts the powers of momenta in the integrand.

 $\sim \dfrac{1}{\not{k}} \sim \dfrac{1}{k}$

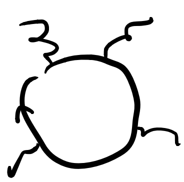
 $\sim \dfrac{1}{k^2}$

$$\int d^dk \sim k^d \sim k^4$$

 $\sim k$ (derivative coupling)

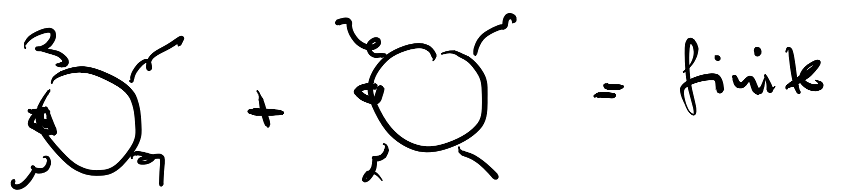
e.g.

Superficial degree of divergence $D=0$.

 $\sim \int d^4k \, \dfrac{1}{k^4} \sim k^0$

... logarithmically divergent

Of course, it can happen that the divergent part vanishes when computing the integral, e.g.



but a diagram can never be more divergent than the superficial degree of divergence introduced above.

When we introduced renormalized parameters, we implicitly assumed that the renormalized Lagrangian has the same structure as the tree-level one, i.e.

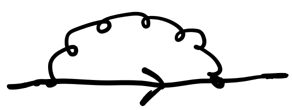
we introduced only one renormalized coupling, despite the ~~fact~~ that there are several terms where g appears

g $\quad$ g $\quad$ g $\quad$ g^2

It is quite natural (~~because~~ of gauge invariance) that this works and one can show that it works to all orders by using Slavnov-Taylor identities which relate different Green's functions.

Let us look at an explicit example.

In the exercise class we computed

 $\equiv \Sigma = \frac{\alpha_s}{4\pi}\frac{C_F}{\varepsilon}\Big[-(1-\xi)\not{p}$

$$+ (4-\xi)\, m_f\Big]$$

$+$ finite

Diagram is $-i\Sigma$

To get the counterterms, we write

$$Z_\psi\, \bar\psi(i\not{\partial} - Z_m m)\psi$$

$$= \bar\psi(i\not{\partial} - m)\psi + \delta_\psi \bar\psi(i\not{\partial} - m)\psi$$

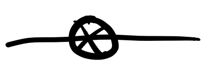
$$- \delta_m m \bar\psi\psi + O(\delta^2)$$

$\propto g^4 \sim \alpha_s^2$

where $$Z_\psi = 1 + \delta_\psi$$
$$Z_m = 1 + \delta_m$$

counter terms, $O(\alpha_s)$

$-i\Sigma = i\not{p}\,\delta_\psi + \dots$

 $= -\not{p}\,\delta_\psi + m_f(\delta_\psi + \delta_m)$

$$\Rightarrow \quad \delta_\psi = \frac{\alpha_s}{4\pi}\left[-\frac{C_F}{\varepsilon}(1-\xi) + c_\psi^{(0)}\right]$$

$$\delta_m = \frac{\alpha_s}{4\pi}\left[-\frac{C_F}{\varepsilon}\,3 + c_m^{(0)}\right]$$

The finiteness of the two-point function fixes the divergent part, but different choices for the finite part are possible. The simplest scheme is Minimal Subtraction (MS), where only the divergent pieces are subtracted.

Proceeding in this way, we now have a finite quark mass $m_q(\mu)$. This quark mass is a parameter in $\mathcal{L}_{QCD}$. It is not directly physical (depends on μ!) but we can compute physical quantities in terms of this mass. This makes talking about quark masses a bit delicate: since we do not have free quarks, there is no such thing as THE quark mass. Instead, we must specify the scheme & scale, when talking about these quantity.

Similarly, by computing the gluon self-energy diagrams and the quark-gluon vertex, one obtains

(*)

$$Z_g = 1 - \frac{\alpha_s}{4\pi}\frac{1}{2\varepsilon}\left(\frac{11}{3}C_F - \frac{4}{3}n_f T_F\right)$$

With this, we have renormalized the theory at one loop. When expressed in terms of the MS parameters $m_s(\mu)$, $\alpha_s(\mu) = g_s^2(\mu)/4\pi$, we obtain finite results for the Green's functions of the theory.

Of course, since these are Green's functions of quarks and gluons, it is not clear at all what their physical significance is. We will come back to this in the second part of the lecture.