Faddeev-Popov Lagrangian

The simplest way to quantize non-abelian gauge theories is the path-integral formalism. The naive form of the path integral is

$$Z = \int \mathcal{D}A_\mu \exp(iS[A]) \qquad (*)$$

$$\prod_x \prod_{\mu,a} dA^a_\mu(x)$$

ill defined because $S[A]$ is gauge invariant: all configurations related by transformation $V(x) = \exp[i\alpha^a(x)t^a]$ give the same value of value of $S[A]$, e.g. all fields:

$$A_\mu = \frac{i}{g} V^\dagger(x)\partial_\mu V(x) \quad \text{have} \quad S[A] = 0$$

In physical expectation values the integral over gauge-equivalent configurations drops out, but this trivial, infinite prefactor makes (*) ill-defined.

In the Standard-Model lecture, the solution to this problem was explained in detail following Faddeev & Popov.

One inserts

Integral over all gauge confs

$$1 = \int \mathcal{D}\alpha \, \delta(G(A_\alpha)) \det\left(\frac{\delta G(A_\alpha)}{\delta \alpha}\right)$$

into the functional integral, where

$$A_\alpha^\mu = e^{i\alpha^a t^a}\left[A^\mu - \frac{i}{g}\partial^\mu\right]e^{-i\alpha^a t^a}$$

The function $G(A_\alpha)$ fixes a gauge but we restore invariance by integrating over all gauges.

Choosing $G(A) = \partial^\mu A^a_\mu - \omega^a$, integrating over $\omega^a(x)$ with Gaussian weight & representing the determinant as a path integral over a Lorentz-scalar Grassman fields η^a & $\bar{\eta}^a$

$$\det\left(\frac{\delta g^a(A)}{\delta \alpha^b}\right) = \int d\bar{\eta} \int d\eta$$

$$\cdot \exp\left(-i \int d^4x \int d^4y \, \bar{\eta}^a(x) \frac{\delta g^a(A(x))}{\delta \alpha^b(y)} \eta^b(y)\right)$$

one finds a form of the Lagrangian which is no longer manifestly gauge invariant, but suitable for perturbative calculations.

We refer to the SM script for a step-by-step derivation of the corresponding Lagrangian. The end result are the following terms, which need to be added to the gauge invariant part of the Lagrangian constructed earlier:

$$
\Delta \mathcal{L}_{FP} = -\frac{1}{2(1-\xi)}\left(\partial^\mu A_\mu^a\right)^2 - \bar{\eta}^a \Box \eta^a + g f_{abc} (\partial^\mu \bar{\eta}^a) \cdot A_\mu^c \eta^b .
$$