1 The Faddeev-Popov Lagrangian

Note: The following discussion is taken from my Standard-Model script.

We now derive the gauge fixing terms needed to quantize the gauge theory. By far the simplest method is to use the path integral, which for a gauge theory naively takes the form

$$\mathcal{Z} = \int \mathcal{D}\mathbf{A}_\mu \, \exp\left(iS[\mathbf{A}_\mu]\right) \quad \text{with} \quad S[\mathbf{A}_\mu] = -\frac{1}{4}\int d^4x \, F^a_{\mu\nu} F^{\mu\nu a} \, . \tag{1}$$

We have left out the fermions, since they do not play a role for the following discussion. The problem with the above expression is that many field configurations lead to exactly the same value of the action. In particular, all pure gauge configurations

$$\mathbf{A}_\mu(x) = -\frac{i}{g}\mathbf{V}(x)\partial_\mu \mathbf{V}^\dagger(x) \tag{2}$$

give a vanishing action, since they can be obtained from $\mathbf{A}_\mu(x) = 0$ with a gauge transformation. For gauge invariant quantities, the integration over physically equivalent gauge field configurations amounts to a trivial but infinite prefactor, so that (1) is ill-defined.[1]

To obtain a meaningful expression, we would like to factor out this integration. Faddeev and Popov [1] came up with a general method to do this.

1.1 A simple example

Before applying their method to the the path integral, let us go over the necessary steps for an ordinary integral. Consider

$$I = \int dx \int dy \, f(x,y) \tag{3}$$

where $f(x,y)$ is a rotation invariant function. The rotation invariance corresponds to the gauge invariance of the path integral integrand. We want to bring this into the form

$$I = \int_0^{2\pi} d\theta \int_0^\infty dr \, F(r) \, , \tag{4}$$

where the first factor is the trivial integration over the symmetry group. For this simple integral, the problem is solved by using spherical coordinates, but for the gauge symmetry we do not know how to make a coordinate transformation in which the trivial part of the integral factors out. Faddeev and Popov provided a general method to achieve this goal and we now illustrate it in our trivial example, before applying exactly the same procedure to the path integral for gauge theory.

[1] The problem does not arise in the discretized version of gauge theories if one works with the link fields $U(y,x)$ instead of the gauge field $\mathbf{A}_\mu(x)$. The link fields $U(y,x)$ are elements of the compact gauge group so that the integration over the symmetry group is finite. In lattice simulations, one is therefore not forced to fix the gauge.

As a first step, we fix a direction (a gauge) by the condition

$$y_\theta = x \sin\theta + y\cos\theta \stackrel{!}{=} 0 \tag{5}$$

However, integrating over only a single direction might be dangerous. To be sure to maintain rotation invariance, we also integrate over all directions in the form

$$\int_0^{2\pi} d\theta \delta(y_\theta) \left| \frac{\partial y_\theta}{\partial\theta} \right| = 2\,. \tag{6}$$

The factor 2 arises because the directions θ and $\theta + \pi$ are equivalent. The jacobian is

$$\left| \frac{\partial y_\theta}{\partial\theta} \right|_{y_\theta=0} = x\cos\theta - y\sin\theta\big|_{y_\theta=0} = \sqrt{x^2+y^2} \tag{7}$$

Now we insert (6) into the original integral

$$I = \int_0^{2\pi} d\theta \int dx \int dy\, \delta(y_\theta)\, \frac{1}{2}\sqrt{x^2+y^2}\, f(x,y) \tag{8}$$

Now comes the crucial step: we rotate our coordinate system by an angle θ so that the new coordinates (x', y') are given by

$$y' = y_\theta \qquad\qquad x' = x\cos\theta - y\sin\theta \tag{9}$$

After this transformation the integrand no longer depends on θ and we have thus factored out the trivial integration over the symmetry group:

$$\begin{aligned} I &= \int_0^{2\pi} d\theta \int dx' \int dy'\, \delta(y')\, \frac{1}{2}\sqrt{x'^2+y'^2}\, f(x',y') \\ &= (2\pi) \int dx' \frac{x'}{2}\, f(x',0) \end{aligned} \tag{10}$$

Note that we have made use of rotation invariance of the integrand to replace $f(x,y) \to f(x',y')$.

1.2 The real thing

We now apply the same technique to the path integral (1). We will consider a general linear gauge fixing functional $G(\mathbf{A})$ and then integrate over all gauges in the form[2]

$$1 = \int \mathcal{D}\boldsymbol{\alpha}\, \delta(G(\mathbf{A}_\alpha)) \det\left(\frac{\delta G(\mathbf{A}_\alpha)}{\delta\boldsymbol{\alpha}} \right) \tag{11}$$

[2]The identity (11) only holds if the solution to the δ-function condition is unique, see (6). It isn't, because of "Gribov copies" but these do not contribute in perturbation theory.

with $\boldsymbol{\alpha} = \alpha_a \mathbf{T}^a$ and where the gauge transformed field is defined as

$$\mathbf{A}^\mu_\alpha = e^{i\boldsymbol{\alpha}} \left[\mathbf{A}^\mu - \frac{i}{g} \partial^\mu \right] e^{-i\boldsymbol{\alpha}} \tag{12}$$

The above two equations are completely analogous to (5) and (6) in our simple example. Note that the jacobian in (11) is independent of $\boldsymbol{\alpha}$ for a linear gauge fixing condition.

We now proceed in the same way as above, by first inserting (11) into the path integral (1) and then performing a variable transformation $\mathbf{A}^\mu \to \mathbf{A}^\mu_\alpha$. The transformation is a shift, followed by a unitary rotation and leaves the measure invariant. Dropping the index α on the gauge field, we arrive at

$$Z = (\int \mathcal{D}\boldsymbol{\alpha}) \times \int \mathcal{D}\mathbf{A}_\mu \, \delta(G(\mathbf{A})) \det \left(\frac{\delta G(\mathbf{A})}{\delta \boldsymbol{\alpha}} \right) \exp \left(iS[\mathbf{A}_\mu] \right) . \tag{13}$$

We have succeeded to factor out the integration over the gauge group, at the price of making the functional integral more complicated. The remaining task is to bring the integral into a form suitable for perturbation theory.

To do so, we choose the gauge fixing condition to have the form

$$G(\mathbf{A}) = g^a(\mathbf{A}) - \omega^a(x) \tag{14}$$

for some function $\omega(x)$. Popular choices for the remainder are

$$g^a(\mathbf{A}) = \partial^\mu A^a_\mu \qquad \text{(“Lorenz gauge”)} \tag{15}$$

$$g^a(\mathbf{A}) = n^\mu A^a_\mu \qquad \text{(“axial gauge”)} \tag{16}$$

The first one is most often adapted, since it does not require the introduction an additional external vector, as is the case for axial gauge. To get rid of the δ-functional, we then integrate over all functions $\omega^a(x)$ with Gaussian weight:

$$\begin{aligned} Z = & \int \mathcal{D}\boldsymbol{\omega} \exp \left(-i \int d^4x \frac{(\omega^a(x))^2}{2(1-\xi)} \right) \\ & \times \int \mathcal{D}\mathbf{A}_\mu \, \delta(g^a(\mathbf{A}) - \omega^a) \det \left(\frac{\delta g^a(\mathbf{A})}{\delta \boldsymbol{\alpha}} \right) \exp \left(iS[\mathbf{A}_\mu] \right) \\ = & \int \mathcal{D}\mathbf{A}_\mu \det \left(\frac{\delta g^a(\mathbf{A})}{\delta \boldsymbol{\alpha}} \right) \exp \left(iS[\mathbf{A}_\mu] - i \int d^4x \frac{1}{2(1-\xi)} (g^a(\mathbf{A}))^2 \right) . \end{aligned} \tag{17}$$

The extra term is precisely what was added to the QED Lagrangian to achieve gauge fixing. The only remaining problem is the presence of the jacobian determinant. To compute the determinant in perturbation theory, we now represent it as an integral over auxiliary Grassman “ghost” fields η^a and $\bar{\eta}^a$:

$$\det \left(\frac{\delta g^a(\mathbf{A})}{\delta \boldsymbol{\alpha}} \right) = \int \mathcal{D}\boldsymbol{\eta} \mathcal{D}\bar{\boldsymbol{\eta}} \exp \left(-i \int d^4x d^4y \, \bar{\eta}^a(x) \, \frac{\delta g^a(\mathbf{A}(x))}{\delta \alpha^b(y)} \eta^b(y) \right) . \tag{18}$$

Note that these fields do not carry a Dirac index and transform as scalars under Lorentz transformations. Since they have the wrong relation between spin and statistics the Feynman-De Witt-Faddeev-Popov ghosts η and $\bar{\eta}$ cannot be interpreted as physical particles. In fact, their role is precisely to cancel the unphysical degrees of freedom in the gauge field $\mathbf{A}_\mu$. However, perturbation theory for these unphysical degrees of freedom works exactly as in the standard case.

Let us now work out the explicit form of the ghost action for Lorentz gauge. We have

$$\begin{aligned}\mathbf{T}^a g^a(\mathbf{A}) = \mathbf{T}^a \partial^\mu A_{\alpha_\mu}^{\ a} &= \partial^\mu e^{i\boldsymbol{\alpha}} \left(\mathbf{A}_\mu - \frac{i}{g}\partial_\mu \right) e^{-i\boldsymbol{\alpha}} \\ &= \partial^\mu \left(\mathbf{A}_\mu - \frac{1}{g}\partial_\mu \boldsymbol{\alpha} + i\left[\boldsymbol{\alpha}, \mathbf{A}_\mu\right] \right) + \mathcal{O}(\alpha^2) \\ &= \mathbf{T}^a \partial^\mu \left(A_\mu^a - \frac{1}{g}\partial_\mu \alpha^a - f_{abc}\alpha^b A_\mu^c \right) + \mathcal{O}(\alpha^2) \end{aligned} \tag{19}$$

The functional derivative thus yields

$$\frac{\delta g^a(\mathbf{A}(x))}{\delta \alpha^b(y)} = -\frac{1}{g}\left(\delta_{ab}\,\Box\ + g f_{abc}\partial^\mu A_\mu^c + g f_{abc} A_\mu^c \partial^\mu\right)\delta^{(4)}(x-y) \tag{20}$$

When inserted into (18), the first term on the right gives a kinetic term for the ghost fields, while the second one gives an interaction between the gauge fields and the ghosts. Because of the prefactor $1/g$, the kinetic term is not properly normalised. One then rescales the ghost fields by a factor $\sqrt{g}$ to bring the action into canonical form.

In an Abelian gauge theory $f_{abc} = 0$. In this case, the ghost fields no longer interact with the gauge fields and can immediately be integrated out so that only the gauge fixing term remains in the action.

Exercise Derive the gauge-fixing Lagrangian in axial gauge $g^a(\mathbf{A}) = n^\mu A_\mu^a$ and derive the associated gauge-boson propagator $G_{\mu\nu}(k)$. Show that for $\xi = 1$ this propagator fulfils $n_\mu G_{\mu\nu}(k) = n_\nu G_{\mu\nu}(k) = 0$ and that this implies that for this choice the ghost fields do not interact. The axial gauge has furthermore the property that

$$\lim_{k^2 \to 0} k^2 k_\mu G_{\mu\nu}(k) = 0\,, \tag{21}$$

so that the unphysical polarisation does not have an associated propagator pole. For this reason, these gauges are also referred to as "physical" gauges. Their disadvantage is the necessity for an external reference vector and the complicated form of the gauge-boson propagator.

2 The Lagrangian for a General Non-Abelian Gauge Theory

Let us summarize what we have found so far: Consider a gauge group $\mathcal{G}$ of "dimension" N (for $SU(n)$: $N \equiv n^2 - 1$), whose N generators, $\mathbf{T}^a$, obey the commutation relations $\left[\mathbf{T}^a, \mathbf{T}^b\right] = i f_{abc}\mathbf{T}^c$, where f_{abc} are called the "structure constants" of the group.

The Lagrangian density for a gauge theory with this group, with a fermion multiplet ψ_i, is given by

$$\mathcal{L} \;=\; -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + i\overline{\psi}\,\left(\gamma^\mu \mathbf{D}_\mu - m\mathbf{I}\right)\psi - \frac{1}{2(1-\xi)}(\partial^\mu A^a_\mu)^2 + \mathcal{L}_{\mathrm{FP}} \tag{22}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f^{abc} A^b_\mu A^c_\nu, \tag{23}$$

$$\mathbf{D}_\mu = \partial_\mu \mathbf{I} + i\,g\,\mathbf{T}^a A^a_\mu \tag{24}$$

and

$$\mathcal{L}_{\mathrm{FP}} \;=\; -\bar{\eta}^a \partial^\mu \partial_\mu \eta^a + g\, f_{acb}(\partial^\mu \bar{\eta}^a)\, A^c_\mu\, \eta^b. \tag{25}$$

References

[1] L.D. Faddeev and V.N. Popov. Feynman Diagrams for the Yang-Mills Field. *Phys.Lett.*, B25:29–30, 1967.