β-Function and Asymptotic Freedom

From the fact that the bare coupling is μ-independent, we can obtain an equation for the μ-dependence of the renormalized coupling constant.

We define the β-function through

$$\boxed{\overbrace{\frac{d}{d\ln\mu}}^{=\mu\frac{d}{d\mu}} \alpha_s(\mu) := \beta(\alpha_s, \varepsilon)}$$

$$\frac{d}{d\ln\mu}\alpha_{s_0} = 0$$

depends on μ via $\alpha_s(\mu)$!

$$= \frac{d}{d\ln\mu} Z_g^2 \mu^{2\varepsilon} \alpha_s(\mu)$$

$$= \left(\frac{d}{d\ln\mu} Z_g^2\right) \mu^{2\varepsilon} \alpha_s(\mu)$$

$$+ 2\varepsilon Z_g^2 \mu^{2\varepsilon} \alpha_s(\mu)$$

$$+ Z_g^2 \mu^{-2\varepsilon} \frac{d}{d\ln\mu} \alpha_s(\mu)$$

d-dim β-function

$$\Longrightarrow \boxed{\beta(\alpha_s, \varepsilon) = -2\varepsilon \alpha_s - 2\alpha_s Z_g^{-1} \frac{d}{d\ln\mu} Z_g} \quad (**)$$

We will now derive a simple

"magic" relation between the β-function and the $\frac{1}{\varepsilon}$-poles in Z_g in the MS scheme.

To do so, we expand

need upper cutoff?

$$\beta(\alpha, \varepsilon) = \overset{d=4}{\beta(\alpha)} + \sum_{n=1}^{\infty} \varepsilon^n \beta^{[n]}(\alpha_s)$$

$$Z_g = 1 + \sum_{k=1}^{\infty} \frac{1}{\varepsilon^k} Z_g^{[-k]}(\alpha_s)$$

MS $\hat{=}$ only poles!

and use that

$$\frac{dZ_g}{d\ln\mu} = \frac{dZ_g}{d\alpha_s}\frac{d\alpha_s}{d\ln\mu} = \frac{dZ_g}{d\alpha_s}\beta(\alpha_s, \varepsilon)$$

With this (**) implies

$$Z_g \, \beta(\alpha_s, \varepsilon) = -2\varepsilon \alpha_s Z_g$$

$$- 2\alpha_s \frac{dZ}{d\alpha_s} \beta(\alpha_s, \varepsilon)$$

Now expand at large ε (!) and compare coefficients. (exercise)

$$\beta^{[1]} = -2\alpha_s + 0$$

Similarly

$$\beta^{[n]} = 0 \quad \text{for} \quad n > 1$$

The $O(1)$ term of the equation yields

$$\beta(\alpha_s) \cancel{- 2\alpha_s Z_g^{[-1]}} = \cancel{-2\alpha_s Z_g^{[-1]}}$$

$$- 2\alpha_s \frac{dZ^{[-1]}}{d\alpha_s} \cdot (-2\alpha_s)$$

$$\boxed{\beta(\alpha_s) = 4\alpha_s^2 \frac{dZ_g^{[-1]}}{d\alpha_s}}$$

"magic relation" (***)

and $\beta(\alpha_s, \varepsilon) = \beta(\alpha_s) - 2\alpha_s \varepsilon$

Note: A similar result can be derived for the μ-dependence of the mass:

$$\frac{d}{d\ln\mu} m(\mu) = \gamma_m \, m(\mu)$$

and

$$\gamma_m = 2\alpha_s \frac{dZ_m^{[-1]}}{d\alpha_s}$$

Using the one-loop result (*) for Z_g yields

$$\beta(\alpha_s) = 4\alpha_s^2\left[-\frac{1}{4\pi}\frac{1}{2}\left(\frac{11}{3}C_F - \frac{4}{3}n_f T_F\right)\right]$$

$$= -2\alpha_s \frac{\alpha_s}{4\pi}\beta_0$$

with $\beta_0 = \frac{11}{3}C_F - \frac{4}{3}n_f T_F$

More generally, one expands

$$\beta(\alpha_s) = -2\alpha_s\left(\beta_0\frac{\alpha_s}{4\pi} + \beta_1\left(\frac{\alpha_s}{4\pi}\right)^2 + \dots\right)$$

In QCD $\beta_0 = \frac{11}{3} \cdot 3 - \frac{4}{3} \cdot 6 \cdot \frac{1}{2} = 7$

so the coupling decreases as μ increases!

It is easy to solve the differential equation using separation of variables. At leading order, we have

$$\frac{d}{d\ln\mu} a_s(\mu) = -2 a_s^2 \beta_0 \quad \text{with } a_s = \frac{\alpha_s(\mu)}{4\pi}.$$

so $$-\frac{da_s}{\beta_0 a_s^2} = 2\, d\ln\mu$$

$$-\int_{a_s(\mu_0)}^{a_s(\mu)} \frac{da_s}{\beta_0 a_s} = 2\int_{\mu_0}^{\mu} \frac{d\mu}{\mu}$$

$$\frac{1}{a_s(\mu)} - \frac{1}{a_s(\mu_0)} = \beta_0 \ln\left(\frac{\mu^2}{\mu_0^2}\right)$$

$$\rightarrow \quad a_s(\mu) = \frac{a_s(\mu_0)}{1 + \beta_0 a_s(\mu_0) \ln\left(\frac{\mu^2}{\mu_0^2}\right)}$$

$$\text{or} \quad \alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \beta_0 \frac{\alpha_s(\mu_0)}{4\pi} \ln\left(\frac{\mu^2}{\mu_0^2}\right)}$$

Observations

* Coupling decreases for large $\mu \gg \mu_0$!

  A logarithmically slow decrease, but asymptotically one ends up with a free theory of quarks & gluons.

  $\leadsto$ Nobel prize for Politzer, Gross & Wilczek

* Coupling increases as $\mu \to 0$.

In fact $\alpha_s(\mu)$ diverges when

$$\frac{\alpha_s(\mu_0)}{4\pi} \beta_0 \ln\left(\frac{\mu_0^2}{\mu^2}\right) = 1 \qquad \text{"Landau pole"}$$

Landau scale

$$\Longleftrightarrow \quad \mu^2 = \Lambda^2 = \mu_0^2 \, e^{-\frac{4\pi}{\beta_0 \alpha_s(\mu_0)}} \approx 150 \text{ MeV}$$

strong coupling at low energies "infrared slavery". Perturbation breaks down at low energies.

* Note: $\alpha_s(\mu)$ is not directly physical. When computing physical quantities at scale Q in perturbation theory, one finds μ independence, up to higher orders, but higher order coefficients involve

logarithms $\ln(\mu^2/Q^2)$. If $\mu \gg Q$ or $\mu \ll Q$ these logarithms become large, which spoils convergence. Need to use $\mu \approx Q$ to get reliable predictions. In practice, one often varies $Q/2 < \mu < 2Q$ to estimate the size of higher-order corrections.

We can thus interpret $\alpha_s(\mu)$ as the coupling strength at an energy scale μ.

* For large energies it makes sense to work with $n_f = 6$, since there are 6 quark flavors. However, at low energies the top quark should not play a role. We'll discuss the issue of heavy flavors in the next chapter.