Transformation of Goldstone fields

Usually fields transform linearly as a representation of the symmetry group G:

$$\vec{\varphi} \xrightarrow{g \in G} M(g)\,\vec{\varphi}$$

However Goldstone bosons are excitations around a vacuum state that is not invariant under the symmetry. We will see that the fields describing them transform nonlinearly. We will analyze a general situation, where a symmetry group G is spontaneously broken to a subgroup

H. There are then $n = n_G - n_H$ Goldstone bosons, which we collect into a vector $\vec{\pi}(x)$. A realization of the group is a mapping

$$\vec{\pi} \xrightarrow{g} \vec{\pi}\,' = \vec{f}(g, \vec{\pi}) \quad \forall g \in G.$$

The mapping must obey composition

$$\vec{f}(g_1 \cdot g_2, \vec{\pi}) = \vec{f}(g_1, f(g_2, \vec{\pi})).$$

Remarkably, this determines $\vec{f}$ essentially uniquely. To see this consider the image of the origin $\vec{f}(g, \vec{\pi} = 0)$. Since the unbroken subgroup H is linearly realized, it maps the origin onto itself

$$\vec{f}(h, 0) = 0 \quad \forall h \in H.$$

Therefore, we have

$$\vec{f}(gh, 0) = \vec{f}(g, 0)$$

So $\vec{f}$ lives on the coset space G/H and provides a map from G/H to the Goldstone fields. This map is also invertible since $\vec{f}(g_1, 0) = \vec{f}(g_2, 0)$ implies that $g_1 H = g_2 H$.

Proof: $\vec{f}(g_1^{-1} g_2, 0) = \vec{f}(g_1^{-1}, \vec{f}(g_2, 0))$

$$= \vec{f}(g_1^{-1}, \vec{f}(g_1, 0)) = \vec{f}(g_1^{-1} g_1, 0)$$

$$= \vec{f}(e, 0) = 0$$

$$\rightarrow g_1^{-1} g_2 \in H \rightarrow g_2 H = g_1 H$$

So the function $\vec{\pi} = \vec{f}(g, 0)$ provides a one-to-one mapping between G/H and values of the field. The transformation of the field follows from the action of g on the coset space. The remaining freedom is the choice of coordinates on G/H. Consider $G = SU_L(n_f) \times SU_R(n_f)$

$$= \{ (V_L, V_R),\ V_{L,R} \in SU(n_f) \}$$

$$H = SU_V(n_f) = \{ (V, V),\ V \in SU(n_f) \}$$

The coset space G/H associated with an element $\tilde{g} \equiv (\tilde{V}_L, \tilde{V}_R)$ is the set

$$\tilde{g}H = \{ (\tilde{V}_L \cdot V, \tilde{V}_R \cdot V), \ V \in SU(n_f) \}$$

To parameterize G/H, we can select one element of each set gH. A simple choice is to write

$$(\tilde{V}_L, \tilde{V}_R) = \underbrace{(1, \tilde{V}_R \tilde{V}_L^{+})}_{\text{representative element}} \cdot \underbrace{(\tilde{V}_L \cdot V, \tilde{V}_L \cdot V)}_{\in H}$$

The matrix $U = V_R V_L^{+}$ parametrizes G/H and transforms as

$$U \xrightarrow{g} V_R U V_L^{+} \quad \text{for } g \equiv (V_L, V_R)$$

All that is left is to parameterize $U(x) \in SU(n_f)$.

The standard parameterization used in the literature is

$$U(x) = \exp\left[i t^a \alpha^a(x) \right]$$

with $\alpha^a = \frac{2\pi^a(x)}{F}$, $a = N_f^2 - 1$. The fields $\pi^a(x)$ are the Goldstone boson fields.

For $n_f = 2$, we have

$$U(x) = \exp\left[i \sigma^a \pi^a / F \right] = \exp\left[\frac{i}{F} \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} \right]$$

In the second step, we have written π^1, π^2, π^3 in terms of linear combinations with definite charge. Let us also anticipate that the constant F introduced to make the exponent dimensionless will correspond to the $\pi^{\pm}$ decay constant.

To see this, we should write down an effective Lagrangian for the field $U(x)$.