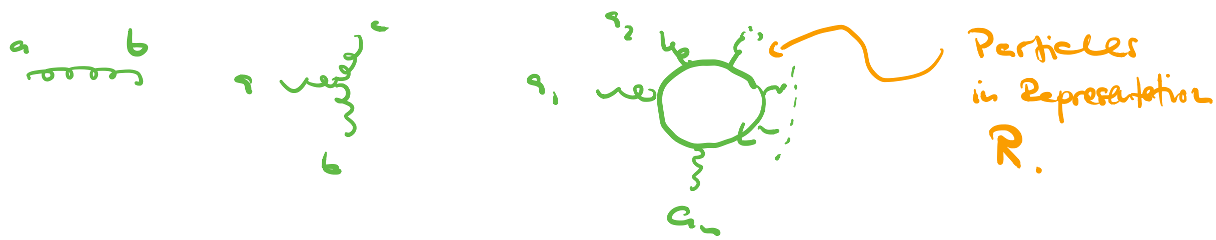
Color factors in Feynman diagrams

To evaluate Feynman diagrams in gauge theories, we need to simplify the color structure in Feynman diagrams. For color-singlet objects we encounter the objects

$$\delta^{ab} \quad , \quad f^{abc} \quad , \quad \mathrm{tr}\left[t_R^{a_1} \cdot t_R^{a_2} \cdot \ldots \cdot t_R^{a_n} \right]$$



in which all indices a, b, c, a_i are connected.

One can express these structures in terms of a few basic invariants of the underlying Lie group, namely the Casimir invariants

$$C_{R,R'}(N) = \sum_{a_1, \ldots, a_n} t_R^{a_1} \cdot t_R^{a_2} \cdot \ldots \cdot t_R^{a_n} \, \mathrm{tr}\left[t_{R'}^{a_1} \cdot \ldots \cdot t_{R'}^{a_n} \right] \qquad (*)$$

One can show that $C_{R,R'}(N)$ commutes with all generators t_R^a. If R is irreducible, this implies that $C_{R,R'}(N)$ is constant on R.

The most important invariant is the quadratic Casimir

$$C_R = C_{R,F}(2)\,\frac{1}{T_F} = t_R^a\, t_R^b \; \mathrm{tr}\left(t^a t^b\right)\cdot\frac{1}{T_F}$$

$$= t_R^b\, t_R^b \,.$$

Let us verify that it commutes with all generators

$$\left[t_R^b \cdot t_R^b\,,\, t^a\right] = t_R^b\, t_R^b\, t_R^a - t_R^a\, t_R^b\, t_R^b$$

$$+ \underbrace{t_R^b\, t_R^a\, t_R^b - t_R^b\, t_R^a\, t_R^b}_{=0}$$

$$= t_R^b\, i f^{bac}\, t_R^c - i f^{abc}\, t_R^c\, t_R^b$$

$$= i f^{abc}\left(-t_R^b\, t_R^c - t_R^c\, t_R^b\right) = 0$$

$\rightsquigarrow C_R \equiv C_R \cdot \mathbb{1}$.

In the exercise class, we have evaluated C_F and C_A for $SU(N)$ and found

$$C_F = \frac{N^2-1}{2N} \quad ; \quad C_A = N$$

It turns out that up to two loops, the quadratic invariants are sufficient to evaluate any diagram. Beyond this higher invariants $C_{R,R'}(N)$ are needed, but not all of these are independent. First of all, it is sufficient to only consider symmetric traces. Antisymmetric pieces can be reduced using the group identity

$$[t_R^a, t_R^b] = i f^{abc} t_R^c .$$

One can also show that of the remaining invariants only r are independent. A detailed, algorithmic way of computing color factors is presented in hep-ph/9802376.

A simple, but less general way of computing color structures in SU(N) is to use the color Fierz identity

$$t^a_{ij}\, t^a_{kl} = \frac{1}{2}\left(\delta_{il}\,\delta_{jk} - \frac{1}{N}\,\delta_{ij}\,\delta_{kl}\right) \qquad (**)$$

and $f^{abc} = 2\,\mathrm{tr}\left([t^a, t^b]\,t^c\right)$. After eliminating the f^{abc} and using (**) all computations boil down to contractions of Kronecker δ's. This can easily be implemented into a computer code.