Feynman rules

We now want to derive the Feynman rules for QCD. Before doing so, it is worth revisiting the derivation for a scalar theory.

Let

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2}_{\mathcal{L}_0} \underbrace{- \frac{\lambda}{4!}\phi^4}_{\mathcal{L}_I}$$

Define

$$\langle \phi_1 \cdots \phi_n \rangle := \langle \phi(x_1) \cdots \phi(x_n) \rangle$$

$$= \frac{1}{Z}\int \mathcal{D}\phi \; \phi(x_1) \cdot \ldots \cdot \phi(x_n)\, e^{i\int d^4x\, \mathcal{L}_0}$$

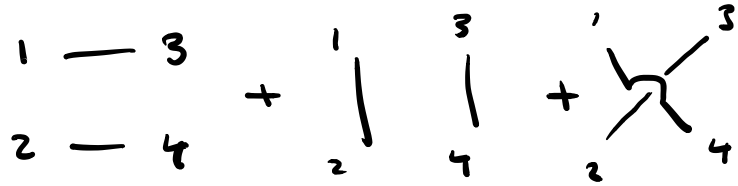
$$= \langle 0 | T[\hat{\phi}(x_1) \cdot \ldots \cdot \hat{\phi}(x_n)] | 0 \rangle$$

Feynman rules are a graphical way of representing Wick's theorem:

$$\langle \phi_1 \cdot \ldots \cdot \phi_n \rangle = \sum_{\text{all pairings}} \underbrace{\langle \phi_{i_1} \phi_{i_2} \rangle}_{\text{free propagator}} \langle \phi_{i_3} \phi_{i_4} \rangle \cdot \ldots \cdot \langle \phi_{i_{n-1}} \phi_{i_n} \rangle$$

e.g.

$\langle \phi_1 \cdots \phi_4 \rangle =$



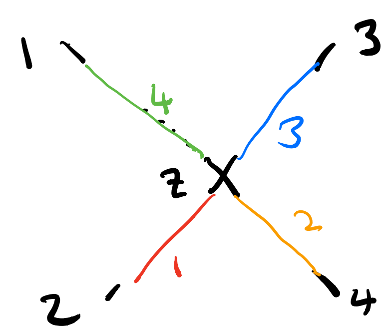
Interacting theory:

$$z \langle \phi_1 \cdot \ldots \cdot \phi_4 \, e^{i \int d^4x \, \mathcal{L}_I} \rangle$$

$$= \langle \phi_1 \cdot \ldots \cdot \phi_4 \rangle - i \frac{\lambda}{4!} \int d^4x \langle \phi_1 \cdot \ldots \cdot \phi_4 \, \phi_z^4 \rangle$$

$$- \frac{1}{2!} \left(\frac{\lambda}{4!} \right)^2 \int d^4x \int d^4y \langle \phi_1 \ldots \phi_4 \, \phi_x^4 \, \phi_y^4 \rangle$$

Number of equivalent contractions at $O(\lambda)$



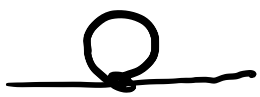
$= 4!$

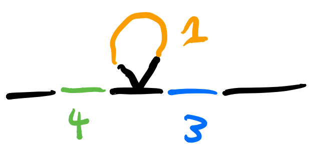
Convention for Feynman rules

A.) Vertex Feynman rule is $-i\frac{\lambda}{4!}$, but we count the number of contractions

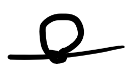
Standard in QCD literature

B.) Feynman rule is $-i\lambda$, but we need to insert a symmetry factor if not all contractions arise.

Example: 

A.) 

$-i\frac{\lambda}{4!}\,4\cdot 3 = -i\frac{\lambda}{2}$

B.) 

$-i\lambda\frac{1}{2}$ ↙ Symmetry factor

Most of the times, one works with Feynman rules in momentum space. To obtain them, one Fourier transforms the Lagrangian

$$-\frac{\lambda}{4!}\int d^4x\,\phi^4(x) = -\frac{\lambda}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}\int d^4x\, e^{-i(k_1+\dots+k_4)x}\,\tilde{\phi}(k_1)\cdot\ldots\cdot\tilde{\phi}(k_4)$$

$$\int_k \equiv \int\frac{d^4k}{(2\pi)^4}$$

$$= -\frac{\lambda}{4!}\int_{k_1}\int_{k_2}\int_{k_3}\int_{k_4}(2\pi)^4\,\delta^{(4)}(k_1+k_2+k_3+k_4)\,\tilde{\phi}(k_1)\cdot\ldots\cdot\tilde{\phi}(k_4)$$

At tree-level, all momentum integrations can be performed trivially. Doing so, leads to momentum conservation at each vertex, plus an overall momentum conservation δ-function. At higher orders integrals over the loop momenta remain.

Let us now go through the procedure in QCD.
The free Lagrangian is

$$\mathcal{L}_0 = \sum_f \bar{\psi}_f(i\not{\partial}-m)\psi_f - \frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2$$

$$- \frac{1}{2(1-\xi)} (\partial^\mu A^a_\mu)^2 \quad - \bar{\eta} \Box \eta .$$

Using integration by part these quadratic parts of the action can be brought into the form $\phi_i D_{ij} \phi_j$, where D_{ij} is a differential operator. The inverse is most easily taken in Fourier space, remembering that $i\partial_\mu \hat{=} k_\mu$.

In this way, one finds

$$\langle \psi^f_{i\alpha}(x) \bar{\psi}^f_{j\beta}(y) \rangle = \int_k \left(\frac{i}{\not{k} - m_f + i\varepsilon} \right)_{\alpha\beta} (\mathbb{1})_{ij} \, e^{-ik(x-y)}$$

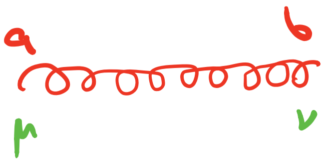
color

Dirac

$$\begin{matrix} \beta & & \alpha \\ y & \longrightarrow & x \end{matrix}$$

Usually the quark color and spin indeces are kept implicit. For the gluon propagator in Fourier space one has

color-diagonal

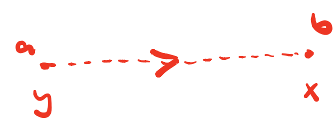


$$\equiv \frac{i}{k^2 + i\varepsilon} \left(-g_{\mu\nu} + \xi \frac{k_\mu k_\nu}{k^2} \right) \delta^{ab}$$

same as γ-propagator

The only new element, compared to QED is the ghost propagator

$$\langle \eta^a(x)\,\bar{\eta}^b(y)\rangle = \int_k \frac{i}{k^2}\,\delta^{ab}\, e^{-ik(x-y)}$$



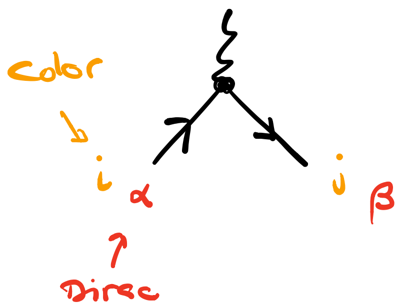
The ghost field is denoted by a dotted line. We put an arrow to indicate particle flow. Note that the ghost field has a bosonic propagator, but anticommutes. This is unphysical, but ok since the role of the ghosts is to cancel unphysical contributions from the gluon field!

Next, let's derive the vertices.

$$\mathcal{L}_{Int} = g A^a_\mu \bar{\psi} \gamma^\mu t^a \psi - g f^{abc} \partial^\beta A^{\alpha,a} A^b_\beta A^c_\alpha$$
$$- \frac{g^2}{4} f^{abe} A^a_\alpha A^b_\beta f^{cde} A^c_\alpha A^d_\beta$$

$$- g f^{abc} \, \bar{\eta}^a \, \partial^\mu A^c_\mu \, \eta^b$$

The first term yields the quark gluon vertex



$$ig \, (\gamma^\mu)_{\beta\alpha} \, (t^a)_{ji}$$

The second term is a three-gluon interaction

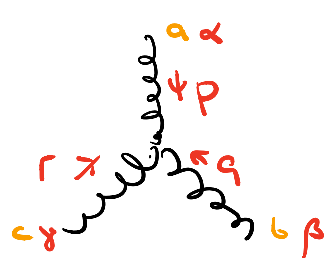
Since we want to seperate the external fields,

we first rewrite

$$-f^{abc} \, (i\partial^\beta A^a_\alpha) \, A^b_\beta \, A^c_\alpha$$

$$= -f^{abc} \, (i\partial^\beta A^a_\alpha) \, A^b_\beta \, A^c_\gamma \, g^{\alpha\gamma}$$

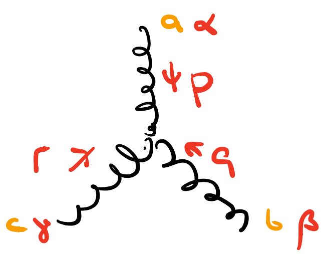
It's Fourier transform is



$$\hat{=} \; - g \, p^\beta g^{\alpha\gamma} f^{abc}$$

$+$ "permutations"

We follow the convention B.) and the Feynman rule is the sum over all 3! permutations of the external gluon fields:



$$= -g\, p^{\beta} g^{\alpha\gamma} f^{abc}$$
$$-g\, q^{\gamma} g^{\beta\alpha} f^{cab}$$
$$-g\, r^{\alpha} g^{\gamma\beta} f^{bca}$$

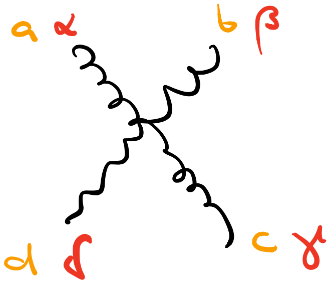
3 × zyklisch

$$-g\, q^{\alpha} g^{\beta\gamma} f^{bac}$$
$$-g\, p^{\gamma} g^{\alpha\beta} f^{cba}$$
$$-g\, r^{\beta} g^{\gamma\alpha} f^{acb}$$

+ (1 ↔ 2) in each case

$$= +g f^{abc}\Big[(p-q)^{\gamma} g^{\alpha\beta} + (q-r)^{\alpha} g^{\beta\gamma} + (r-p)^{\beta} g^{\alpha\gamma}\Big]$$

Similarly, the gluon term is obtained as



$$\hat{=} -\frac{ig^2}{4} f^{abe} f^{cde} g_{\alpha\gamma} g_{\beta\delta}$$

$$+ \text{"permutations"}$$

In this case, we have $4!$ permutations of which always 4 are equivalent. This gives

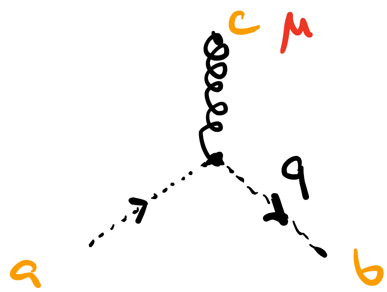
$$-ig^2 f^{abe} f^{cde} \left(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\delta} \right)$$

$$-ig^2 f^{ace} f^{bde} \left(g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\beta\gamma} \right)$$

$$-ig^2 f^{ade} f^{bce} \left(g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\gamma} g_{\beta\delta} \right)$$

for the full Feynman rule.

Finally, we need the ghost - gluon vertex



$$\hat{=} -g\, q^{\mu} f^{abc}$$

(we have integrated by part to move the derivative to the outgoing ghost line and use an outgoing momentum!)