Effective Lagrangian

Our goal is now to construct

$\mathcal{L}_{eff} \equiv \mathcal{L}_{eff}(\mathcal{U})$ invariant under

$\mathcal{U} \longrightarrow V_R \mathcal{U} V_L^{\dagger}$. As usual, we will

order the terms in $\mathcal{L}_{eff}$ by dimension,

but since $\mathcal{U}(x)$ is dimensionless this

amounts to an expansion in derivatives:

$$\mathcal{L}_{eff} = \underbrace{f_0(\mathcal{U})}_{O(1)} + \underbrace{f_1(\mathcal{U}) \Box \mathcal{U}}_{O(p^2)} + \underbrace{f_2(\mathcal{U})\, \partial_\mu \mathcal{U}\, \partial^\mu \mathcal{U}}_{O(p^2)} + O(p^4)$$

will discuss later, how indices are contracted

- symmetry implies $f_0(U) = f_0(V_R U V_L^\dagger)$.

Choose $V_R = \mathbb{1}$, $V_L = U$: $f_0(U) = f(1) = \text{const}$.

- Partielle integration:

$$\int d^4x \, f_1(U) \Box U = - \int d^4x \, f_1'(U) \partial_\mu U \partial^\mu U$$

$$\Rightarrow \mathcal{L}_{eff} \hat{=} f_2(U) \partial_\mu U \partial^\mu U$$

$$= \tilde{f}_2(U) \Delta_\mu \Delta^\mu$$

with $\Delta_\mu = \partial_\mu U \cdot U^\dagger$. Note that Δ_μ is invariant under V_L transformations. and transforms as $\Delta_\mu \to V_R \Delta_\mu V_R^\dagger$.

$$\mathcal{L}_{eff} = \tilde{f}(U V_L^\dagger) \Delta_\mu \Delta^\mu \underset{V_L = U}{=} \tilde{f}(1) \Delta_\mu \Delta^\mu.$$

The Lagrangian must also be invariant under V_R transformations. The only possibility is

$$\mathcal{L}_{eff} = \hat{f} \cdot \mathrm{tr}[\Delta_\mu \Delta^\mu].$$

Note that

$$\mathrm{tr}[\Delta_\mu \Delta^\mu] = \mathrm{tr}[(\partial_\mu U) \cdot U^+ (\partial^\mu U) \cdot U^+]$$

$$= - \mathrm{tr}[(\partial_\mu U) U^+ U \cdot (\partial^\mu U^+)] = - \mathrm{tr}[\partial_\mu U \partial^\mu U^+]$$

$\partial_\mu U U^+ = \partial_\mu \mathbb{1} = 0$

In summary, we have

$$\mathcal{L}_{eff} = \frac{F^2}{4} \mathrm{tr}[\partial_\mu U \partial^\mu U^+] + O(p^4)$$

The prefactor has been chosen to get a canonically normalized kinetic term for $\vec{\pi}$.

To see this, we expand

$$U(x) = \exp\left[\frac{i}{F}\vec{\sigma}\cdot\vec{\pi}\right] = \mathbb{1} + \frac{i}{F}\vec{\pi}\vec{\sigma} - \frac{1}{2F^2}\vec{\pi}^2\cdot\mathbb{1}$$

$$+ \dots$$

for $SU(2)$, which yields

$$\mathcal{L}_{eff} = \frac{1}{2}\partial_\mu\vec{\pi}\,\partial^\mu\vec{\pi} + \frac{1}{6F^2}\left[\left(\vec{\pi}\partial_\mu\vec{\pi}\right)^2 - \vec{\pi}^2\left(\partial_\mu\vec{\pi}\right)^2\right]$$

$$+ O(\pi^6)$$ (exercise)

Properties of $\mathcal{L}_{eff}$:

1.) Interactions with arbitrary many π's!

2.) One parameter F determines all these interactions (up to p^2-suppressed terms!)

3.) Derivative couplings: the interactions go to zero, when the energies go to zero.

The above Lagrangian is valid in the strict chiral limit $m_f \to 0$. We now want to add small non-zero quark masses

$$\mathcal{L}_m = -\bar{\Psi}_R \mathcal{M} \Psi_L - \bar{\Psi}_L \mathcal{M}^\dagger \Psi_R$$

with $\mathcal{M} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$.

To do so, we'll use the following trick. We treat $\mathcal{M}$ as an extern source which transforms as $\mathcal{M} \to V_R \mathcal{M} V_L^\dagger$.

For such a source $\mathcal{L}_{QCD}$ remains chirally invariant. We then construct

$$\mathcal{L}_{eff} = \mathcal{L}_{eff}(U, m)$$

to be invariant as well. Expanding in powers of m, the lowest invariant term is

$$\mathcal{L}_{s.b.} = \frac{F^2 B_0}{2} \, tr\left[m U^\dagger + m^\dagger U \right]$$

After inserting the quark mass matrix, this term generates masses for the pions. For $SU(2)$

$$\mathcal{L}_{s.b.} = \frac{F^2 B_0}{2} \, tr\left[m \right] \left(-\frac{1}{F^2} \vec{\pi}^2 \right)$$

$$= - \frac{B_0}{2} (m_u + m_d) \vec{\pi}^2 \hat{=} - \frac{M_\pi^2}{2} \vec{\pi}^2$$

$\rightsquigarrow$ Up to higher order corrections, all π's have the same mass, proportional to the sum of the quark masses.

To relate B_0 to a QCD matrix element we treat $m = [m(x)]_{ij}$ as an external source and compute

$$\frac{1}{i}\frac{\delta}{\delta m_{ij}(x)} Z_{QCD} = -\langle 0 | \bar{q}_{L,i}(x)\, q_{R,j}(x) + \bar{q}_{R_j}(x)\, q_{L_i}(x) | 0 \rangle$$

$$\frac{1}{i}\frac{\delta}{\delta m_{ij}(x)} Z_{eff} = \frac{F^2 B_0}{2} \langle 0 | U^\dagger_{ji}(x) + U_{ij}(x) | 0 \rangle$$

The classical action is minimized by $\vec{\pi}(x) = 0$, $U = \mathbb{1}$. Up to pion loop corrections, we thus have

$$F^2 B \delta_{ij} = - \langle 0 | \bar{q}_{L,i} q_{R,j} + \bar{q}_{R,j} q_{L,i} | 0 \rangle$$

$$\rightsquigarrow F^2 B_0 = - \langle 0 | \bar{u} u | 0 \rangle = - \langle 0 | \bar{d} d | 0 \rangle$$

$\rightsquigarrow$ B_0 corresponds to the quark condensate.

in the limit $m_f \to 0$.

$$\rightsquigarrow M_\pi^2 = \underbrace{(m_u + m_d)}_{\text{explicit breaking}} \left(\frac{\overbrace{- \langle 0 | \bar{u} u | 0 \rangle}^{\text{spontaneous breaking}}}{F^2} + O(m_f^2) \right)$$

Since $p_\pi^2 = M_\pi^2 \propto m_f$, we count the quark masses as $O(p^2)$.

For SU(2) all pions have the same mass because

$$\mathcal{U}(x) = 1 + \frac{i}{F} 2t^a\pi^a - \frac{1}{2F^2} \underbrace{4\, t^a\pi^a t^b\pi^b}_{2\{t^a,\, t^b\}\pi^a\pi^b}$$

and in SU(2)

$$2\{t^a, t^b\} = \frac{1}{2}\{\sigma^a, \sigma^b\} = \delta^{ab}.$$

In SU(3) the commutator has a nontrivial structure

leading to

$$M_\pi^2 = (m_u + m_d) B_0 + O(m_q^2)$$

$$M_{K^\pm}^2 = (m_u + m_s) B_0 + O(m_q^2)$$

$$M_K^2 = M_{\bar{K}}^2 = (m_d + m_s) B_0 + O(m_q^2)$$

$$M_\eta^2 = \frac{1}{3}(m_u + m_d + 4m_s) B_0$$

(Gell-Mann, Oakes, Renner '68)

$$\leadsto \quad M_\pi^2 - 4M_K^2 + 3M_\eta^2 = 0 + O(m_q^2)$$

Gell-Mann-Okubo formula

To understand how the mesons interact with photons, W- and Z-bosons, one can introduce external sources with the appropriate quantum numbers and then construct $\mathcal{L}_{eff}$ in the presence of the sources. (This is good enough for a discussion at the classical level: to include QED loop effects, we would need a dynamical photon field.)

We thus write $\mathcal{L}_{QCD} = \mathcal{L}_0 + \mathcal{L}_1$

$$\mathcal{L}_0 = \bar{\psi} i \not{D} \psi - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a}$$

← no masses! can use s^a to have them

classical sources

$$\mathcal{L}_1 = v^a_\mu V^\mu_a + a^a_\mu A^\mu_a - s^a S_a - p^a P_a$$

with

acts in flavor space!

$$V^\mu_a = \bar{\psi} \gamma^\mu t^a \psi \qquad A^\mu_a = \bar{\psi} \gamma^\mu \gamma^5 t^a \psi$$

$$S_a = \bar{\psi} t^a \psi \; ; \qquad P_a = \bar{\psi} i \gamma^5 t^a \psi$$

To also include singlets, we introduce $t^0 \propto \mathbb{1}$.

To construct $\mathcal{L}_{eff}$ one uses that $\mathcal{L}_{QCD}$ becomes invariant under local chiral transformations

$$\psi_L(x) \longrightarrow V_L(x)\,\psi_L(x) \;;\; \psi_R(x) \longrightarrow V_R(x)\, q_R(x)$$

provided the external fields transform like gauge fields. Writing $v_\mu = v_\mu^a t^a$, we need

$$r_\mu = v_\mu + a_\mu \longrightarrow V_R (v_\mu + a_\mu) V_R^+ - i(\partial_\mu V_R) V_R^+$$

$$\ell_\mu = v_\mu - a_\mu \longrightarrow V_L\, \ell_\mu V_L^+ - i(\partial_\mu V_L) V_L^+$$

$$s + ip \longrightarrow V_R (s+ip) V_L^+$$

With this, it becomes easy to construct $\mathcal{L}_{eff}$. One simply replaces the ordinary derivative by the covariant one:

count as $O(p)$

$$i D_\mu U = i\partial_\mu U + r_\mu U - U \ell_\mu$$

(exercise: show that $i D_\mu U \longrightarrow V_R\, i D_\mu U\, V_L^+$)

The lowest order gauge inv. $\mathcal{L}_{eff}$ is

$$\mathcal{L}_{eff} = \frac{F^2}{4} \mathrm{tr}\left[(D_\mu U)(D^\mu U^\dagger)\right]$$

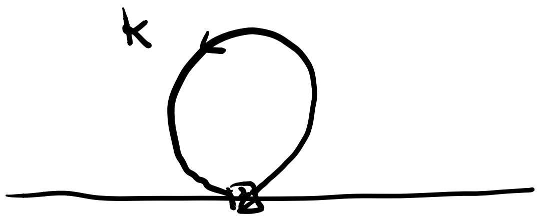
$$+ \frac{F^2 B}{2} \mathrm{tr}\left[\chi U^\dagger + U \chi^\dagger\right] + O(p^4)$$

with $\chi = s + ip$

contains quark masses, $O(p^2)$

The Lagrangian at $O(p^4)$ was constructed by Gasser & Leutwyler '84. For SU(3) it contains 12 parameters ("low-energy constants"). For SU(2), there are 10. Gasser and Leutwyler also showed that the one-loop diagrams of $\mathcal{L}_{eff}^{(2)}$ contribute at $O(p^4)$ and can be renormalized with the parameters of $\mathcal{L}_{eff}^{(4)}$.

Let us look at an example:

$\Sigma(p^2) \sim$  $\sim \int d^4k \, \frac{i}{k^2} \, \overset{O(k^2)}{V(p,k)}$

vertex from $\mathcal{L}^{(2)}$: $O(p^2)$

$\sim O(p^4)$

Using the same methods we used to compute D in QCD, one can show that all one-loop diagrams from $\mathcal{L}^{(2)}_{eff}$ contribute at $O(p^4)$. And since $\mathcal{L}^{(4)}_{eff}$ is the most general Lagrangian at this order, it must be possible to absorb the divergences into its parameters. More generally, at any order p^n only a finite number of parameters arise and all divergences can be absorbed into a redefinition of these parameters.

Since the parameters (F, B_0, ...) describe low-energy properties of QCD, one either has to extract them using a fit to data, or compute them with a nonperturbative method such as lattice QCD.

Unfortunately the number of parameters rapidly grows

p^2	p^4	p^6	p^8	p^{10}	p^{12}
2	12	117	1959	45171	1170086
	Gasser & Leutwyler '84	Fearing Scherer '96 Bijnens, Colangelo, Ecker '99	"Nils et al." 1810.06834		see 2008.01239

With this proliferation of parameters, one starts to loose predictivity at higher orders.

The other limitation is that the effective theory ~~breaks~~ down once the energies become large enough that heavier particles (e.g. the ρ-meson with $M_\rho \approx 770\,\mathrm{GeV}$) can be produced.

To extend the predictions to higher energies CHPT is often combined with dispersive methods.

To finish the construction of $\mathcal{L}_{eff}$, we briefly touch on an interesting complication. The Lagrangian $\mathcal{L}_{eff}$ is invariant under the full set of chiral transformations, but QCD is not invariant due to anomalies associated with the fermion determinant. To reproduce the behavior of QCD, one needs to add an additional term which reproduces the change of the QCD partition function. This term is called the Wess-Zumino-Witten term $\mathcal{L}_{WZW}$.

The full effective theory Lagrangian is then

$$\mathcal{L}_{eff} = \mathcal{L}_{inv} + \mathcal{L}_{WZW}$$

starts at $O(p^4)$, no parameter at this order.