7. Advanced methods for multi-jet processes

Thanks to its high energy and luminosity, the LHC produces multi-jet events with quite high multiplicities. There are measurements of

$$pp \to n \text{ jets}$$

$$pp \to W/Z + m \text{ jets}$$

for $n = 8$ with only $36\,pb^{-1}$. With the 2012 data, one should be able to probe $n \approx 11-12$!

Also $m = 4$ was measured with only $36\,pb^{-1}$.

These multijet processes are interesting from a QCD perspective, but they also form backgrounds to New Physics processes, where a heavy particle decays into a cascade of lighter ones.

Feynman diagram computations become quickly cumbersome (and then impossible) as the number of external legs increases, even at tree level.

The number of diagrams rises quickly. For pure gluon amplitudes, one has

n	4	5	6	7	8	9	10
#diags	4	25	220	2485	34'300	559'405	10'525'900

So computing 8-jet production ($n=10$) diagrammatically is out of the question. A second challenge is that one needs to integrate over the final state phase space, which has a high dimension.

Miraculously, however, the result for the amplitudes can take a very simple form in certain cases. For so-called MHV (maximally helicity violating) amplitudes, Parke and Taylor '86 found an extremely simple form. They considered n-gluon amplitudes and found "experimentally" *

$$\mathcal{M}_c(1^+, 2^+, \dots, n^+) = 0 \quad \text{(all helicities positive)}$$

(helicity)

$$\mathcal{M}_c(1^+, \dots, i^-, \dots, n^+) = 0 \quad \text{(one negative hel.)}$$

$$\mathcal{M}_c(1^+, \dots, i^-, \dots, j^-, \dots, n^+)$$

$$= g_s^{n-2} \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle (n-1)\, n \rangle \langle n\, 1 \rangle} \; ; \quad \text{(MHV)}$$

$$\langle ij \rangle = \sum_s \bar{u}(p_i, s) \tfrac{1}{2}(1+\gamma_5)\, u(p_j, s)$$

$\ulcorner$ $\mathcal{M}_c$ is "color-ordered". The full amplitude is obtained as

$$\mathcal{M} = \sum_{\sigma \in S^n / \mathbb{Z}_n} \mathrm{tr}\{ t^{a_{\sigma(1)}} \cdot \dots \cdot t^{a_{\sigma(n)}} \} * \mathcal{M}_c(\sigma(1), \dots, \sigma(n))$$

$\llcorner$

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* They computed analytically for low n and checked the formula numerically for $n \leq 6$, but could not prove it.

It is very remarkable that the gazillion Feynman diagrams combine to this one term and it makes it obvious that Feynman diagrams are not the optimal representation ...

While Parke and Taylor were not able to generalize their results to arbitrary helicity configurations, this has been achieved in a series of papers in 2004, who discovered a set of simple on-shell recursion relations among scattering amplitudes, the BCFW recursion relations (Britto, Cachazo and Feng hep-th/0412308, + Witten hep-th/0501052).

In the following we will briefly discuss the spinor-helicity formalism and the BCFW relation.

After this, we turn to loop computations.

Also here, there has been a breakthrough:

Processes such as $W+3j$, $W+4j$ and even $W+5$ jets have been computed at NLO.

With a purely diagrammatic approach, such computations would be impossible.

The key to these computations were unitarity methods, which allow one to construct loop amplitudes from tree-level results. Efficient tree-level methods then translate into efficient loop computations.

This lecture is not enough to cover the new methods in detail. There are several recent reviews (see the slides for references).

We will restrict ourselves to 3 topics:

1.) an introduction to the spinor-helicity formalism,

2.) the BCFW recursion relation,

3.) loop diagrams: generalized unitarity

Even with this restriction, we will only scratch the surface, so the reader interested in learning more should consult the review articles cited in the accompanying slides.

7.1. Spinor-helicity formalism

Massless particles can be labeled by their helicities. The solutions of the massless Dirac equation are $\left[P_{L/R} = \frac{1}{2}(1 \mp \gamma_5) \right]$

$$u_-(p) = P_L u_-(p) \qquad \text{negative helicity (left handed)}$$

$$u_+(p) = P_R u_+(p) \qquad \text{positive helicity.}$$

Massless antifermions fullfill the same equation, so the above solutions also work for anti-fermions, so $u_-(p)$ describes either an incoming, left-handed fermion, or an out-going, right-handed anti-fermion if $p^0 < 0$.

We now introduce the bra-ket notation

$$u_+(p) = p\rangle \left[\equiv \lambda_\alpha \right] \qquad \bar{u}_-(p) = \langle p$$

$$u_-(p) = p] \left[\equiv \tilde{\lambda}_{\dot{\alpha}} \right] \qquad \bar{u}_+(p) = [p$$

$$\langle p\, q\rangle = \bar{u}_-(p)\, u_+(q)$$

$$\langle p\, q] = \bar{u}_-(p)\, u_-(q) = \bar{u}_-(p)\, \overbrace{\tfrac{1}{2}(1+\gamma_5)\,\tfrac{1}{2}(1-\gamma_5)}^{\frac{1}{4}(1-\gamma_5^2)=0}\, u_-(p)$$

$$= 0$$

Note that

$$p\rangle[p = u_+(p)\,\bar{u}_+(p) = \not{p}\,\frac{1-\gamma_5}{2}$$

$$p]\langle p = u_-(p)\,\bar{u}_-(p) = \not{p}\,\frac{1+\gamma_5}{2}$$

(Follows from $\sum_s u(p,s)\,\bar{u}(p,s) = \not{p}$ after multiplying with $P_{L/R}$.)

Some useful properties:

$$\langle p\, q\rangle[q\, p\rangle = \mathrm{tr}\left[\not{q}\,\frac{1-\gamma_5}{2}\,\not{p}\right]$$

$$= \mathrm{tr}\left[\not{q}\not{p}\,\frac{1+\gamma_5}{2}\right] = 2p\cdot q$$

$$\langle p q\rangle^* = [q p]$$

$$\langle p q\rangle = -\langle q p\rangle \quad ; \quad [p q] = -[q p]$$

Three more useful relations are

$$\langle p \gamma^\mu q] = [q \gamma^\mu p\rangle$$

$$\langle p \gamma^\mu q] \langle k \gamma_\mu l] = 2 \langle pk\rangle [lq]$$

"Fierz"

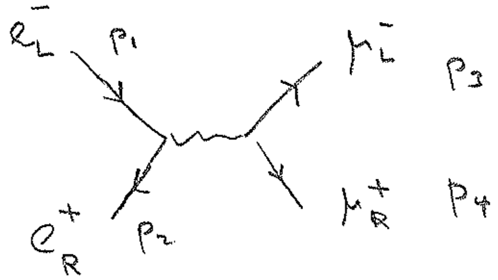
$$\langle p \gamma^\mu q] [k \gamma_\mu l\rangle = 2 \langle pl\rangle [kq]$$

$$\langle ij\rangle\langle kl\rangle + \langle ik\rangle\langle lj\rangle + \langle il\rangle\langle jk\rangle = 0$$

"Schouten"

(and the same with [ ] ...)

Now we can compute:



$$i\mathcal{M} = (-ie)^2 \, \bar{u}_-(2) \gamma^\mu u_-(1) \; \bar{u}_-(3) \gamma_\mu u_-(4) \; \frac{-i}{q^2}$$

$$= \frac{ie^2}{q^2} \langle 2 \gamma^\mu 1] \langle 3 \gamma_\mu 4]$$

$$= \frac{2ie^2}{q^2} \langle 23\rangle [14]$$

Let's square the amplitude:

$$|i\mathcal{M}|^2 = \frac{4e^4}{(q^2)^2} \underbrace{\langle 2 3\rangle [3 2]}_{\underbrace{2 p_2 \cdot p_3}_{= 2 p_1 \cdot p_4}} \underbrace{[1 4]\langle 4 1\rangle}_{2 p_1 \cdot p_4}$$

$$= 4e^4 \frac{u^2}{s^2}$$

Also the polarization vectors for massless gauge bosons can be written in terms of spinors:

$$\varepsilon_+^{*\mu}(k) = -\varepsilon_-^{\mu}(k) = \frac{1}{\sqrt{2}} \frac{\langle r \gamma^\mu k]}{\langle r k\rangle},$$

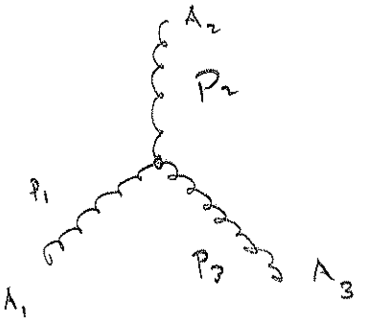
where r^μ is a massless reference vector, with $k \cdot r \neq 0$.

One can check that the vectors fullfill the usual properties of polarization vectors.

often a smart choice of r^μ will simplify computations. Furthermore, for processes with multiple gauge bosons, we can choose different vectors for each boson.

With the polarization vectors at hand, we can now compute n-gluon amplitudes. Lets compute $\mathcal{M}(1^-, 2^-, 3^+)$. This seems useless, since it is impossible to fulfill the momentum conservation constraint for three massless on-shell momenta (an on-shell photon cannot decay into e^+e^-, etc.). However, it is possible with complex momenta. This observation lies at the heart of the BCFW recursion relations and is also essential for the OPP method.

So let's be brave and go ahead...



momenta outgoing

$$= (-i) g_s f^{A_1 A_2 A_3} \Big[(p_1 - p_2)^\gamma g^{\alpha\beta} + (p_2 - p_3)^\alpha g^{\beta\gamma} + (p_3 - p_1)^\beta g^{\gamma\alpha} \Big] \varepsilon_-^{*\alpha}(p_1)\, \varepsilon_-^{*\beta}(p_2)\, \varepsilon_+^{*\gamma}(p_3)$$

$$= i\mathcal{M}(1^-, 2^-, 3^+)$$

We take all particles as outgoing and choose

$$\varepsilon_+^*(3) = \frac{1}{\sqrt{2}} \frac{\langle r \gamma^\mu 3]}{\langle r 3 \rangle}$$

$$\varepsilon_-^*(i) = \frac{-1}{\sqrt{2}} \frac{[r \gamma^\mu i \rangle}{[r i]} \qquad \text{for } i = 1, 2.$$

$$\Rightarrow \quad \varepsilon_-^*(1) \cdot \varepsilon_-^*(2) = 0$$

$$\varepsilon_-^*(i) \cdot \varepsilon_+^*(3) = -\frac{1}{2} \frac{\langle r \gamma^\mu 3] [r \gamma^\mu i \rangle}{\langle r 3 \rangle [r i]}$$

$$= - \frac{\langle r i \rangle [r 3]}{\langle r 3 \rangle [r i]}$$

$$i\mathcal{M}(1^-,2^-,3^+) = -g_s f^{A_1A_2A_3}$$

$$* \Bigg[+\frac{1}{\sqrt{2}} \frac{[r(\overbrace{2}^{=-1-3}-3)1\rangle}{[r1]} \cdot \frac{\langle r2\rangle[r3]}{\langle r3\rangle[r2]}$$

$$+\frac{1}{\sqrt{2}} \frac{[r(3\overbrace{-1}^{=+2+3})2\rangle}{[r2]} \cdot \frac{\langle r1\rangle[r3]}{\langle r3\rangle[r1]} \Bigg]$$

$$= -\frac{g_s}{\sqrt{2}} f^{A_1A_2A_3} \frac{[r3]}{[r1][r2]\langle r3\rangle} \Big\{ -2[r3]\langle 31\rangle\langle r2\rangle$$

$$+ 2[r3]\overbrace{\langle 32\rangle}^{=-\langle 31\rangle}\langle r1\rangle \Big\}$$

$$= -\sqrt{2}\, g_s f^{A_1A_2A_3} \frac{[r3]^2 \overbrace{\langle 13\rangle}^{=\langle 12\rangle}}{[r1][r2]\langle r3\rangle} \Big(\overbrace{\langle r2\rangle + \langle r1\rangle}^{=\langle r3\rangle} \Big)$$

$$= -\sqrt{2}\, g_s f^{A_1A_2A_3} \frac{[r3]^2\langle 12\rangle}{[r1][r2]}.$$

This can be simplified further by multiplying with $\frac{\langle 12\rangle^2}{\langle 12\rangle^2}$.

$$[r1][r2]\langle 12\rangle^2 = [r1]\langle 12\rangle(-1)[r2]\langle 21\rangle$$

$$= [r|\overset{-2-3}{1}|2\rangle\,(-1)\,[r|\overset{-3-1}{2}|1\rangle$$

$$= (+1)[r3]\langle 23\rangle(+1)[r3]\langle 31\rangle$$

$\Rightarrow i\mathcal{M}(1^-, 2^-, 3^+)$

$$= -\sqrt{2}\, g_s\, f^{A_1A_2A_3}\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

Note that

$$f^{ABC} = -\frac{i}{\sqrt{2}}\operatorname{tr}\{t^A t^B t^C\} + \frac{i}{\sqrt{2}}\operatorname{tr}\{t^B t^A t^C\}$$

So we find

$$\mathcal{M}_c(1^-, 2^-, 3^+) = g\,\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 31\rangle}$$

for the color-ordered amplitude.

7.2. The BCFW recursion relation

Let us now come back to the kinematics. Since

we have $$p_1 + p_2 + p_3 = 0$$

$$\rightarrow \quad s_{12} = (p_1+p_2)^2 = p_3^2 = 0$$

$$s_{12} = \langle 12\rangle [21]$$

For real momenta $[21] = \langle 12\rangle^*$

$$\rightarrow \quad s_{12} = |\langle 12\rangle|^2 = 0 \quad \rightsquigarrow \quad \langle 12\rangle = 0.$$

So one would conclude that all the spinor products (and the underlying momenta) must vanish.

For two momenta i and j, BCFW define

$$\hat{i}\rangle = i\rangle \qquad \hat{i}] = i] + z\, j]$$

$$\hat{j}\rangle = j\rangle - z\, i\rangle \qquad \hat{j}] = j]$$

for some complex number z.

The associated momenta are

$$\hat{i}\rangle[\hat{i} = i\rangle[i + z\, i\rangle[j$$

$$\hat{j}\rangle[\hat{j} = j\rangle[j - z\, i\rangle[j$$

so that momentum conservation is respected and also the on-shell conditions are fulfilled since

$$p_{\hat{i}}^2 = \langle \hat{i}\hat{i}\rangle[\hat{i}\hat{i}] = \underbrace{\langle i i\rangle}_{=0}[\hat{i}\hat{i}]$$

$$p_i^2 = \langle \hat{j}\hat{j}\rangle \underbrace{[\hat{j}\hat{j}]}_{=[jj]} = 0$$

Let us now consider an n-point amplitude $\mathcal{M}_c$ and then perform the above shift:

$$M(z) = \mathcal{M}_c(1, \dots, \hat{i}, \dots, \hat{j}, \dots, n)$$

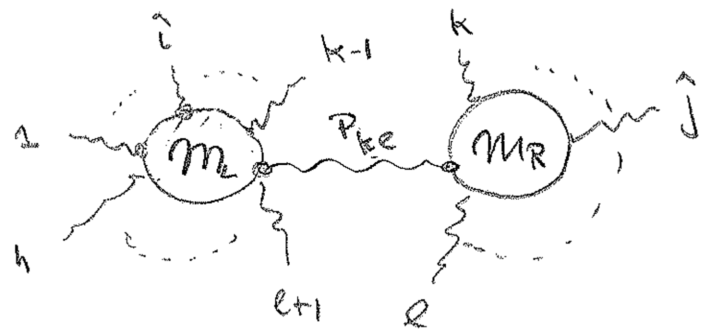
Then consider the integral around a large circle in the complex plane:

$$\oint \frac{dz}{z} M(z)$$

The integral can be evaluated with the residue theorem since $M(z)$ is a rational function of z. For the helicity assignements $(i,j) = (-,+), (-,-), (+,+)$ one can show* that $M(z) \sim \frac{1}{z}$ for $z \to \infty$. In this case

$$\frac{1}{2\pi i} \oint \frac{dz}{z} M(z) = 0 = M(0) + \sum \text{"Residues of propagator poles"}$$

The form of these is as follows



Poles arise when the internal propagator goes on shell:

$$P_{k\ell}^2(z) = \left(\sum_{m=k}^{\ell} p_m\right)^2 = 0$$

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* By counting powers of z in Feynman diagrams

Note that $\hat{i}$ and $\hat{j}$ have to be on opposite sides of the internal propagator which will go on shell. We then have

$$P^2_{k\ell}(z) = P^2_{k\ell}(0) - z \langle i \left(\sum_{m=k}^{\ell} \not{p}_m \right) j]$$

The pole is at

$$z = z_{k\ell} = \frac{P^2_{k\ell}(0)}{\langle i \left(\sum_{m=k}^{\ell} \not{p}_m \right) j]}$$

The residue of this pole is

$$\frac{1}{z_{k\ell}} \cdot \mathcal{M}^L_c \cdot \frac{1}{-\langle i (\Sigma \not{p}_m) j]} \mathcal{M}^R_c = \mathcal{M}_L \frac{-1}{P^2_{k\ell}(0)} \mathcal{M}_R$$

(see the previous page for $\mathcal{M}_L$ and $\mathcal{M}_R$)

The sum of all these residues must be equal to the original amplitude $\mathcal{M}_c(0)$

So we get:

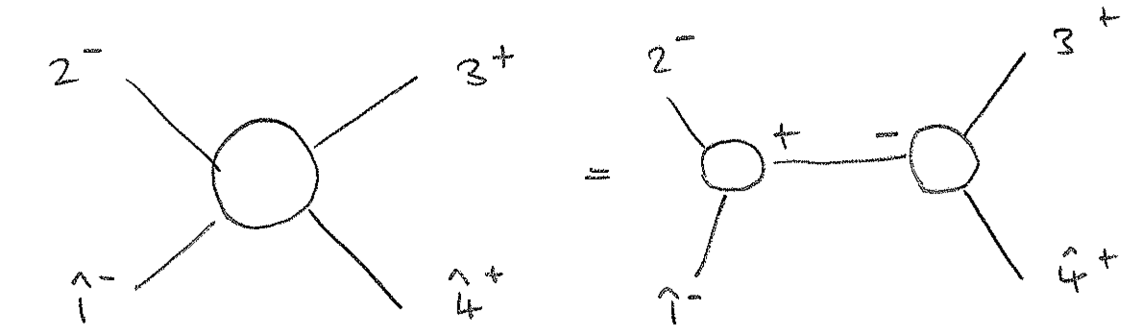
$$\mathcal{M}_c(0) = \sum_{k,e} \sum_{h} \frac{\mathcal{M}_c^{L,h}(z_{ke})\, \mathcal{M}_c^{R,-h}(z_{ke})}{P_{ke}^2(0)}$$

↑ helicity of the intermediate gluon

This is the BCFW recursion relation. It expresses the n-point amplitude as a sum of on-shell amplitudes with fewer legs, evaluated at shifted, complex momenta.

To prove it, one has to show that the amplitude $\mathcal{M}(z)$ indeed vanishes for $z \to \infty$, for appropriate $(-+, --, ++)$ choices of the helicities of the shifted momenta.

Let's look at a simple example:



shift: $\hat{1}] = 1] + z\, 4]$

$4\rangle = 4\rangle - z\, 1\rangle$

BCFW:

$$\mathcal{M} = g \underbrace{\frac{\langle \hat{1} 2\rangle^4}{\langle \hat{1} 2\rangle \langle 2 \hat{P}\rangle \langle \hat{P} \hat{1}\rangle}}_{\mathcal{M}(1^-,\, 2^-,\, P^+)} \frac{1}{S_{12}} \cdot (-g) \underbrace{\frac{[3 \hat{4}]^4}{[34][\hat{4}\hat{P}][\hat{P}3]}}_{\mathcal{M}(3^+,\, 4^+,\, -\hat{P}^-)}$$

$\langle \hat{1} 2\rangle = \langle 12\rangle$; $[3\hat{4}] = [34]$

$$\hat{P}\rangle[\hat{P} = -3\rangle[3 - 4\rangle[4 + z\, 1\rangle[4$$

$$\leadsto \langle 1\hat{P}\rangle[\hat{P}3] = -\langle 14\rangle[43]$$

$$\langle 2\hat{P}\rangle[\hat{P}4] = -\langle 23\rangle[34]$$

Therefore

$$\mathcal{M} = -g^2 \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\cancel{[34]}} \cdot \frac{\overbrace{1}^{s_{12}=s_{34}}}{\langle 34\rangle\cancel{[43]}} \frac{\cancel{[34]}^3}{\langle 14\rangle\cancel{[43]}}$$

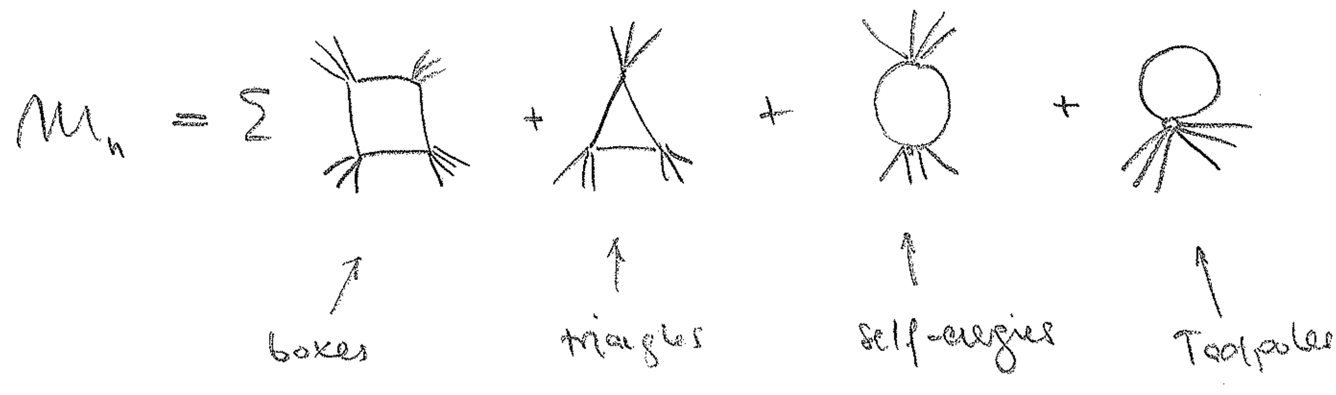
$$= g^2 \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}$$

We recover the ~~Parke~~ Parke-Taylor result for the four-gluon amplitude! It is not difficult to inductively prove the Parke-Taylor formula. Also in the induction step only a single contribution arises and its evaluation is analogous to the above computation.

7.3. Loop diagrams: Generalized unitarity

All the problems arising at the tree-level for a large number of external legs are even more pronounced at the loop level: one ends up with a large number of terms and encounters large cancellations between diagrams and numerical instabilities in certain regions of phase space.

On the other hand, all basic one-loop integrals are known. One has



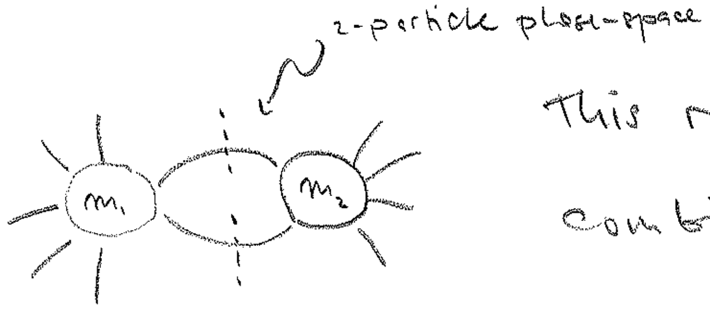
So all that's needed is a determination of the coefficients of the various scalar integrals.

There is a standard technique to reduce one-loop tensor integrals to this basic set of scalar integrals: the Passarino-Veltmann reduction.
However, for many legs the technique becomes cumbersome and it leads to numerical instabilities.
E.g. for a six-point function there are 56 scalar integrals and to get the coefficients one has to invert a 56×56 matrix.

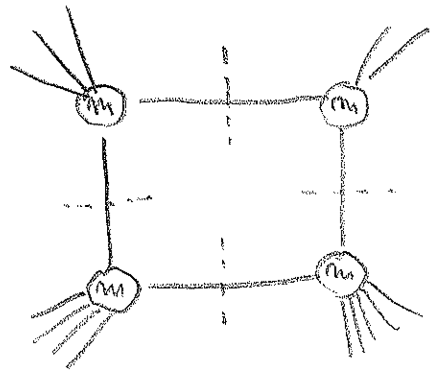
One way to get information about the coefficients is to cut propagators by replacing

$$\frac{1}{p^2 - m^2 + i\varepsilon} \to i\,\Theta(p^0)\,\delta(p^2 - m^2)$$

After cutting two propagators, a loop integral turns into a phase-space integral over tree level amplitudes.



This relates linear combinations of box-, triangle- and self-energy coefficients to tree amplitudes. Britto, Cachazo and Feng '04 pointed out that one can use a quadruple cut to immediately obtain the box-integral coefficients:



The four cuts induce four δ-functions so there are no integrations left. It turns out that one cannot satisfy all four conditions with real momentum, but complex solutions always exist.

By considering all possible quadruple cuts, one obtains all box coefficients. Then one considers

$$\mathcal{M}_n - \sum$$ 

All that's left are triangles, self-energies and tadpoles. The triangles can be isolated by applying triple cuts, etc.

The only contributions one does not get using this method are the so-called rational terms, which arise from divergent loop integrals. In dimensional regularization, these arise when a $\frac{1}{\varepsilon}$ divergence multiplies an $O(\varepsilon)$ term in the numerator. As long as the entire computation is carried out in d dimensions, all terms could be obtained from cuts, but for numerical computations that is impractical.

Giele, Kunszt and Melnikov '08 have shown that one can get the full information by keeping the loop integration d-dimensional, but evaluating the amplitudes with integer spin dimensions $d_s = 5$ and $d_s = 7$. We will not discuss this further, but it means that these are now efficient numerical methods, which determine the coefficients of the loop integrals numerically, in terms of tree-level diagrams (which in turn are obtained by means of recursion relations).

As a proof of principle, Giele and Zanderighi '08 have evaluated one-loop amplitudes with up to twenty external gluons. More importantly, these methods have lead to NLO computations for

$W + 3j$, $W + 4j$, $W + 5j$; $WW + 2j$, $Z + 4j$, $e^+e^- \to 7j$ ...

Also, these techniques can be used to automate one-loop computations and there are several groups who are working on this and have presented first results during the past year.