5. Parton distribution functions

In the last lecture, we have defined jet observables which are infrared finite and can be computed in perturbation theory. There is one last subtlety we need to address in order to have all the pieces in place to systematically compute hadron collider observables. It is related to the Parton Distribution Functions (PDFs) we encountered in our computation of the DIS ($e^- p \to e^- X$) cross section.

We expect that an IR safe cross section at a hadron collider to be given by

$$\sigma(Q) = \sum_{i,j=q,\bar{q},g} \int_0^1 dx_1 \int_0^1 dx_2 \; f_{i/N_1}(x_1, \mu_f)\, f_{j/N_2}(x_2, \mu_f)$$

$$\cdot\; \hat{\sigma}_{ij}(x_1, x_2, Q, \mu_f) \qquad (*)$$

We have assumed that we collide two hadrons N_1 and N_2. Q stands for the kinematic variables the cross section depends on (e.g. p_T of a jet, jet radius R, angle ϕ of some jet, ...). μ_f is the factorization scale, which we explain below. Roughly speaking, it fixes the boundary between the PDFs and the partonic scattering cross section $\hat{\sigma}(x_1, x_2, Q, \mu_f)$: partons with $p_T > \mu_f$ are part of $\hat{\sigma}$, while partons with $p_T \leq \mu_f$ are part of the PDFs.

To discuss this issue in more detail, we first need to properly define the PDFs.

The formula (*) is called a factorization theorem. Strong arguments that it holds for $N_1 + N_1 \to V + X$ (where $V = \gamma^*, W^\pm, Z$) were given by Collins, Soper and Sterman in the 80s. A modern framework to derive such theorems is Soft-Collinear Effective Theory.

5.1. Definition of the PDFs

Let us introduce light-cone reference vectors along the beam axis:

$$n^\mu = (1, 0, 0, 1)$$
$$\bar{n}^\mu = (1, 0, 0, -1)$$

So that $n^2 = \bar{n}^2 = 1$ and $n \cdot \bar{n} = 2$. Every vector can be split into

$$V^\mu = n \cdot V \frac{\bar{n}^\mu}{2} + \bar{n} \cdot V \frac{n^\mu}{2} + V_\perp^\mu$$

Let us consider a nucleon N flying along the z-axis. Neglecting the mass, it's momentum is $P^\mu = \bar{n} \cdot P \frac{n^\mu}{2}$.

The quark PDF is defined as

$$f^{(0)}_{q/N}(\xi) = \frac{1}{2\pi} \int dt \, e^{-i\xi t \bar{n} \cdot P}$$
$$\times \frac{1}{2} \sum_s \langle N(P,s) | \bar{\psi}(t\bar{n}) \frac{\bar{n}\!\!\!/}{2} [t\bar{n}, 0] \psi(0) | N(P,s) \rangle$$

ψ is the quark field for quark flavor q.

$[t\bar{n}, 0]$ is a Wilson line from $0 \ldots t\bar{n}$, which ensures that the definition is gauge invariant.

The Wilson line is

$$[tn, 0] = \mathbb{P} \exp\left[ig \int_0^t dt' \, \bar{n}\cdot A^A(t'\bar{n})\, t^A \right]$$

$\nearrow$ Path-ordering: later $\equiv$ left.

Under a gauge transformation $\psi(x) \to V(x)\psi(x)$

$$[tn, 0] \longrightarrow V(tn)\,[tn, 0]\, V^{+}(0).$$

To make contact with the naive probability interpretation ("$f_{q/N}(\xi)$ is the probability to find a quark with momentum ξP inside N") we now rewrite f by inserting a complete set of states:

$$f_{q/N}(\xi) = \sum_X \frac{1}{2\pi} \int dt\, e^{-i\xi t\bar{n}\cdot P} \, \frac{1}{2} \bar{n}_{\alpha\beta}$$

$$\underbrace{e^{it\bar{n}\cdot \mathbb{P}} \, \bar{\psi}(0)\, e^{-it\bar{n}\cdot\mathbb{P}}}$$

$$* \langle N(P) |\, \bar{\psi}_\alpha(t\bar{n}) \,| X\rangle\langle X |\, \psi_\beta(0) \,| N(P) \rangle$$

$$= \sum_X \frac{1}{2\pi} \int dt\, e^{-i\xi t\bar{n}\cdot P} \, e^{it(\bar{n}\cdot P - \bar{n}\cdot p_X)} \, \frac{1}{2} \bar{\not{n}}_{\alpha\beta}$$

$$* \langle N(P) |\, \bar{\psi}_\alpha(0) \,| X\rangle\langle X |\, \psi_\beta(0) \,| N(P) \rangle$$

$$= \sum_X \delta(\bar{n}\cdot P - \bar{n}\cdot p_X - \xi\, \bar{n}\cdot P)\, \langle N(P) |\, \bar{\psi}_\alpha(0) \,| X\rangle\langle X |\, \psi_\beta(0) \,| N(P) \rangle$$

$$* \frac{1}{2} \bar{\not{n}}_{\alpha\beta}$$

The Dirac structure $\frac{1}{2}\bar{n}\!\!\!/_{\alpha\beta}$ projects out the leading-power components of the quark field. Ignoring this structure, we have

$$f_{q/N}(\xi) \sim \sum_X \delta((1-\xi)\bar{n}P - \bar{n}\cdot P_X)$$
$$\cdot \left| \langle X | \psi(0) | N(P) \rangle \right|^2$$

i.e. $f_{q/N}(\xi)$ is the probability that the quark field carries away longitudinal momentum $\xi \bar{n}\cdot P$. Note that it can carry arbitrary transverse momentum!

There are two related problems:

1.) Since the transverse momentum can be arbitrary large $f_{q/N}$ suffers from UV divergences.

2.) Since the transverse momentum of partons in $\hat{\sigma}$ can be arbitrary small, $\hat{\sigma}$ suffers from IR divergences.

The solution is to introduce a scale Λ_f: A UV cut off for f and an IR cut off for $\hat{\sigma}$. The scale would Λ_f would properly separate the different contributions.

Instead of a hard cut off it is simpler to work with dimensional regularization and subtract the divergences in the $\overline{MS}$ scheme.

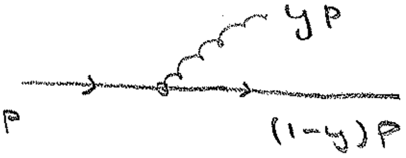
bare renormalized

$$f_i^{(0)}(x, \varepsilon) = \int_x^1 \frac{dy}{y} \underbrace{Z_{ij}(y, \varepsilon, \mu_f)}_{\text{UV div's.}} f_j\left(\frac{x}{y}, \mu_f\right)$$

$d = 4 - 2\varepsilon$

$$Z(y, \varepsilon, \mu_f) = \delta_{ij}\, \delta(1-y) + \frac{\alpha_s(\mu_f)}{\pi} P_{i\leftarrow j}(y) \cdot \frac{1}{\varepsilon}$$

$$+ \dots$$

The $P_{i\leftarrow j}(y)$ are called Altarelli-Parisi splitting functions. $P_{q\leftarrow q} \sim$ 

The Z-factor which governs the UV divergences of $f_i^{(0)}(x)$ is the same as the one governing the IR divergences of $\hat{\sigma}^{(0)}$. We can absorb it into $\hat{\sigma}_{ij}^{(0)}$ after wich we obtain a finite, but μ_f dependent cross section $\hat{\sigma}_{ij}(x_1, x_2, Q, \mu_f)$.

Since $f_i^{(0)}$ is μ_f independent, one obtains an evolution equation (the DGLAP equation)

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$$\frac{d}{d\ln\mu_f^2} f_i(x, \mu_f) = \int_x^1 \frac{dy}{y} \frac{\alpha_s(\mu_f)}{\pi} P_{i\leftarrow j}(y)\, f_i\!\left(\frac{x}{y}, \mu_f\right)$$

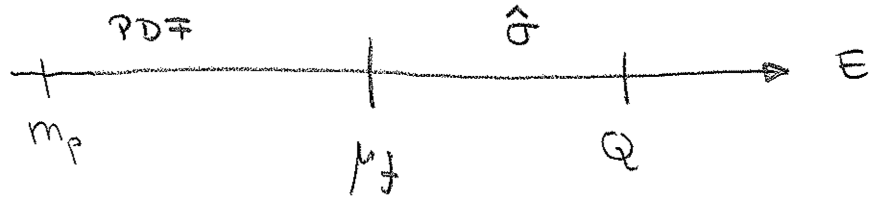
$$+ \mathcal{O}(\alpha_s^2).$$

To see this, note that $\frac{d\alpha_s(\mu)}{d\ln\mu} = \beta(\alpha_s, \varepsilon) = \beta(\alpha_s) - 2\varepsilon\alpha_s$.

In the Z-factor, the ε-dependent piece needs to be kept because it is a divergent quantity.

What is the appropriate value for μ_f ?

From a physics point of view a value in between the non-perturbative scale $\Lambda_{QCD} \sim m_p$ and the high-energy scale seems most appropriate



since the scale separates the two contributions.

On purely theoretical grounds any value of μ_f is fine, since the hadronic cross section $\sigma(Q)$ μ_f independent. However, since we truncate the perturbative expansion of $\hat{\sigma}$ at some fixed order, we should choose μ_f so that the expansion is well behaved. For $\mu_f \ll Q$ logarithmic terms of the form

$$\alpha_s^n(\mu_f)\, \ln^n\left(\frac{Q}{\mu_f}\right)$$

spoil the perturbative expansion. For this reason the default scale choice is usually taken to be $\mu_f = Q$. To get an estimate of the missing higher-order corrections, one usually varies the scale by a factor of 2.

The logarithmic dependence of the PDFs on $\mu_f \approx Q$ has interesting phenomenological consequences. In our parton model calculation we found that the structure functions of DIS were Q-independent. In QCD we obtain logarithmic dependence

$$F_2(x, Q^2) = \sum_{i=q,\bar{q}} e_q^2 \, f_q(x, \overset{\mu_f}{\overset{\downarrow}{Q}})$$

The predicted "scaling violations" are indeed observed in the data.

5.2. Determination of the PDFs

The basic strategy is to extract the PDFs by computing a set of observables and then fitting the predictions to data.

The set includes:

DIS : $e^- N \to e^- X$ $(N = p, d, \dots)$

Drell-Yan: $p\bar{p} \to \gamma^*/Z + X \to e^+e^- + X$
(or pp)

2-jet prod.: $p\bar{p} \to 2\,\text{jets}$ (not in all fits)

One starts with a parameterization of the PDFs at some reference scale μ_0:

$$f_{i/N}(x, \mu_0) = A\, x^{a_i}(1-x)^{b_i}\left(1 + c_i\sqrt{x} + d_i\, x\right)$$

for small and hard-to-determine PDFs fewer parameters are used ($c_i = 0$; $d_i = 0$). Often the number of PDFs is reduced making assumptions such as $f_{s/N}(x,\mu_0) = f_{\bar{s}/N}(x,\mu_0)$

The PDFs are then evolved numerically from μ_0 to μ_f, as appropriate for a given process.

Then the parameters are fitted.

Modern PDF fits take into account the experimental uncertainties and provide a way to obtain uncertainties on the PDFs. This is usually done by providing a "central value" PDF set, together with additional (40 – 1000) error PDFs.

Potential problems:

1.) Parametrization bias?

2.) Non-Gaussianities?

3.) Theoretical uncertainties.

The fits do not take into account theory uncertainties, but different fits are performed using LO, NLO or NNLO predictions: LO PDFs, NLO PDFs, NNLO PDFs.

The fits are performed by a number of different groups, for example

CTEQ (now CT)

MSTW (formerly MRST)

ABKM (formerly Alekhin)

NNPDF

While the first three of these perform χ^2 fits to fixed parametrizations, the last one tries to address problems 1.) and 2.) by using neural networks to parametrize the PDFs and by generating statistical ensembles of PDFs instead of relying on a fit.

LHAPDF provides a unified interface to access all modern PDF sets.