3. Higher-order corrections and infrared safety

We have seen in the last chapter that the lowest order predictions for

$$e^+e^- \to \text{hadrons}$$

$$e^- p \to e^- + \text{hadrons}$$

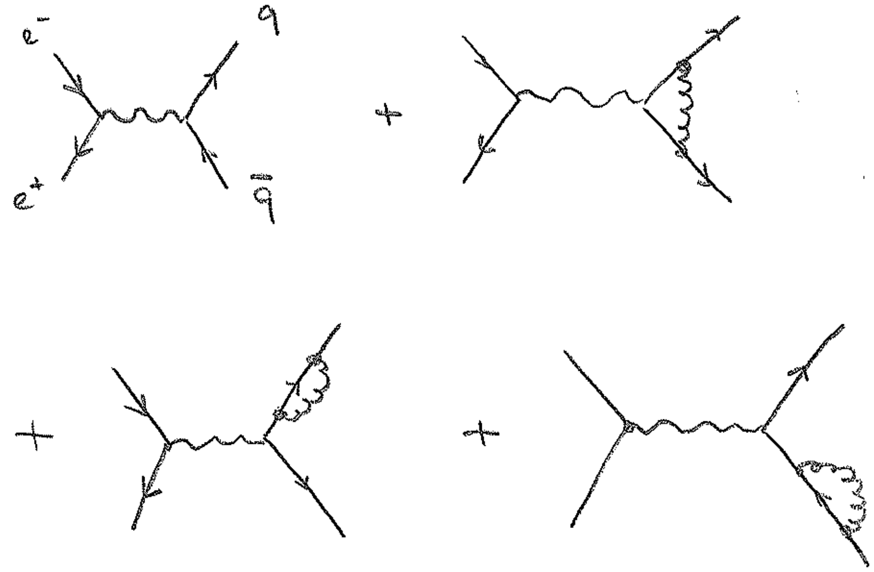
compare very well to data.

It is interesting to ask, whether it is possible to also obtain predictions for less inclusive observables, where we don't just sum over all hadrons in the final state. The answer is yes, for suitably defined observables. To understand what "suitably defined" means, we now compute the perturbative corrections to $e^+e^- \to$ hadrons. At order g_s^2, two processes are possible:

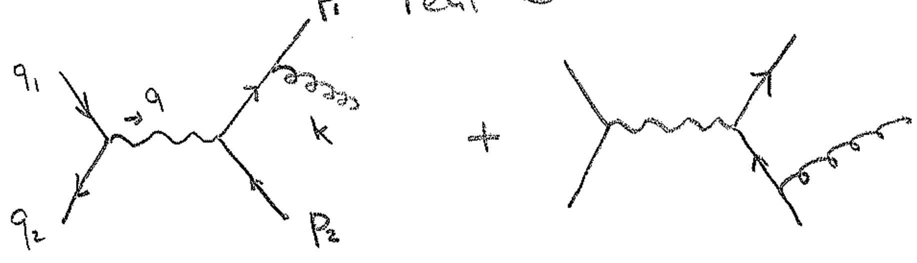
A.) $e^+e^- \to q\bar{q}$

B.) $e^+e^- \to q\bar{q}g$

Virtual corrections

$\mathcal{M}_A =$ 

real emission corrections

$\mathcal{M}_B =$ 

And $\sigma(e^+e^- \to X) = \sigma(e^+e^- \to q\bar{q}) + \sigma(e^+e^- \to q\bar{q}g)$
$+ O(g_s^4)$

Let us first consider the real-emission corrections.

Averaging over initial state spins and summing over the final state spins gives

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_B|^2 = \frac{16\pi}{Q^2}\,\sigma^{(0)}_{q\bar{q}}\;\overset{\frac{N_c^2-1}{2N_c} = 4/3}{C_F}\, g_s^2$$

$$* \;\frac{(p_1\cdot k)^2 + (p_2\cdot k)^2 + q^2\, p_1\cdot p_2}{p_1\cdot k \;\; p_2\cdot k}$$

$|M_3|^2$ diverges when $p_1 \cdot k \to 0$ or $p_2 \cdot k \to 0$.

The denominators arise from the intermediate quark propagators. The singularity arises when the gluon momentum k^μ vanishes (soft limit) or becomes collinear to the quark momenta p_1 or p_2.

A convenient choice of variables for the phase space is y_1, y_2 and y_3:

$$(p_1 + p_2)^2 = y_3 \, q^2 \quad ; \quad (p_1 + k) = y_2 \, q^2$$

$$(p_2 + k)^2 = y_1 \, q^2$$

These have a simple interpretation: In the CMS

$$y_i = 1 - \frac{2E_i}{Q} = 0 \ldots 1 \qquad Q = \sqrt{q^2}$$

and $y_1 + y_2 + y_3 = 3 - \frac{2Q}{Q} = 1$.

Then rewrite $\int dPS_3 = \frac{q^2}{128\pi^3} \int_0^1 dy_1 \int_0^{1-y_1} dy_2$

(we derive this later on page 3.5)

The cross section becomes:

$$\sigma_{q\bar{q}g} = \sigma^{(0)}_{q\bar{q}} \frac{\alpha_s C_F}{4\pi} \int_0^1 dy_1 \int_0^{1-y_1} dy_2 \frac{4y_3 + 2(y_1^2 + y_2^2)}{y_1 y_2}$$

This is ill-defined, because of the soft and collinear singularities! Similarly, the loop integrals will suffer from such singularities.

In order to check whether the cross section itself is OK, we need to regularize both the real and the virtual corrections. One way to do it, would be to give the gluon a small mass λ, and then compute

$$\sigma(e^+e^- \to \text{hadrons}) = \lim_{\lambda \to 0} \left[\sigma_{q\bar{q}} + \sigma_{q\bar{q}g} \right] + O(\alpha_s^2)$$

However $\lambda \neq 0$ destroys gauge invariance. A less intuitive, but technically superior way is to use dimensional regularization.

To understand how this works, we now evaluate the three-particle phase-space in d dimensions.

$$PS_3 = \int \frac{d^{d-1}p_1}{(2\pi)^{d-1}\, 2E_1} \int \frac{d^{d-1}p_2}{(2\pi)^{d-1}\, 2E_2} \int \frac{d^{d-1}k}{(2\pi)^{d-1}\, 2E_3} (2\pi)^d \delta^{(d)}(q-p_1-p_2-k)$$

$$= \frac{1}{8(2\pi)^{2d-3}} \int \frac{d^{d-1}p_1}{E_1} \int \frac{d^{d-1}p_2}{E_2} \frac{1}{E_3} \delta(Q - E_1 - E_2 - E_3)$$

$$\left[\; q^\mu = (Q, 0, \ldots, 0) \qquad E_i = |\vec{p}_i| \right.$$

$$E_3^2 = E_1^2 + E_2^2 + 2E_1E_2 \cos\theta$$

angle between $\vec{p}_1$ and $\vec{p}_2$

$$= \frac{1}{8(2\pi)^{2d-3}} \int_0^{Q/2} dE_1\, E_1^{d-3} \underbrace{\int d\Omega_{d-1}}_{\text{surface of unit sphere}} \int_0^{Q/2} dE_2\, E_2^{d-3} \int_0^{\pi} d\theta\, |\sin(\theta)|^{d-3}$$

$$* \int d\Omega_{d-2}\, \frac{1}{E_3} \delta(Q - E_1 - E_2 - E_3)$$

[set $E_1 = (1-y_1)Q/2$ $\quad E_2 = (1-y_2)Q/2$. Integrate over θ to eliminate δ-function.

$$PS_3 = \frac{\Omega_{d-2}\,\Omega_{d-1}}{8\,(2\pi)^{2d-3}}\; 4^{-1+\varepsilon}\, Q^{2-4\varepsilon} \int_0^1 dy_1 \int_0^{1-y_1} dy_2$$

$$* \left(y_1 y_2 y_3\right)^{-\varepsilon} \quad , \text{where } \; y_3 = 1 - y_1 - y_2$$
$$d = 4 - 2\varepsilon$$

$$\Omega_d = \frac{2\,\pi^{d/2}}{\Gamma(d/2)} \qquad \left[\begin{array}{l} \Omega_2 = 2\pi , \\ \Omega_3 = 4\pi , \ldots \end{array}\right.$$

$$PS_3 = \frac{Q^2}{128\,\pi^3} \left(\frac{4\pi}{Q^2}\right)^{2\varepsilon} \frac{1}{\Gamma(2-2\varepsilon)} \cdot \int_0^1 dy_1 \int_0^{1-y_1} dy_2 \left(y_1 y_2 y_3\right)^{-\varepsilon} .$$

Integrals such as

$$\int_0^1 dy\,(y)^{-\varepsilon}\,\frac{1}{y} = -\frac{1}{\varepsilon}$$

are well defined as long as $\varepsilon < 0$, i.e. $d > 4$.

For $d \to 4$ we encounter a singularity.

By performing the phase-space as well as the loop integrations in d dimensions, the soft and collinear divergences are regularized.

If the entire computation is performed in d dimensions, the real emission correction becomes

$$\sigma_{q\bar{q}g} = \sigma_{q\bar{q}}^{(0)} \frac{\alpha_s C_F}{4\pi\,\Gamma(1-\varepsilon)} \left(\frac{Q^2}{4\pi\mu^2}\right)^{-\varepsilon} \int_0^1 dy_1 \int_0^{1-y_1} dy_2 \,(y_1 y_2 y_3)^{-\varepsilon}$$

$$* \frac{4(y_3 - y_1 y_2 \cdot \varepsilon) + 2(1-\varepsilon)(y_1^2 + y_2^2)}{y_1 y_2}$$

The integrations over y_1 & y_2 can be carried out analytically. Expanding around $d = 4 - 2\varepsilon$, one then obtains

$$\sigma_{q\bar{q}g} = \sigma_{q\bar{q}}^{(0)} \frac{\alpha_s C_F}{4\pi} \left(\frac{Q^2}{4\pi e^{-\gamma_E}\mu^2}\right)^{-\varepsilon} \cdot \left\{ \frac{4}{\varepsilon^2} + \frac{6}{\varepsilon} + 19 - \frac{7\pi^2}{3} + O(\varepsilon) \right\}$$

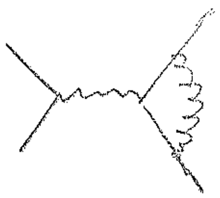
$\uparrow$

divergence in both integrations, from $y_1 \approx y_2 \approx 0$

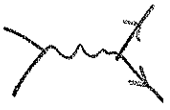
Next, one has to evaluate the virtual corrections in d-dimensions.

$$= C \cdot (-p^2)^{-\varepsilon} = 0 \quad \text{for } p^2 = 0,$$

so only



is nonvanishing.

$$\sigma_{q\bar{q}} = \Big|$$  $$+$$  $$\Big|^2 = |A^{(0)} + A^{(1)}|^2$$

$$= |A^{(0)}|^2 + A^{(0)} A^{*(1)} + A^{(1)} \underset{\uparrow \; A^{(0)*} = A^{(0)}}{A^{(0)}} + O(\alpha_s^2)$$

$$= A^{(0)} \cdot 2\mathrm{Re}\left[A^{(1)}\right] + O(\alpha_s^2)$$

$$= \sigma^{(0)}_{q\bar{q}} \, \frac{\alpha_s C_F}{4\pi} \left(\frac{Q^2}{4\pi e^{-\gamma_E} \mu^2} \right)^{-\varepsilon} \left[-\frac{4}{\varepsilon^2} - \frac{6}{\varepsilon} - 16 + \frac{7\pi^2}{3} \right]$$

$$\Longrightarrow \sigma(e^+e^- \to \text{hadrons}) = \sigma_{q\bar{q}} + \sigma_{q\bar{q}g} + O(\alpha_s^2)$$

$$= \sigma^{(0)}_{q\bar{q}} \left(1 + 3 C_F \frac{\alpha_s}{4\pi} \right)$$

We obtain a finite and well defined result!

Note that the same problems with soft and collinear divergences also arise in massless QED. The problem is that we cannot distinguish an "electron" from an "electron + soft photons". So any sensible QED observable must include some soft photon radiation. We can also not distinguish a "massless electron" from a "massless electron + collinear photon".

So any sensible observable must include collinear radiation.

The total cross section we just computed includes both types of radiation and is therefore well-defined order by order in perturbation theory.

In the next lecture, we will study jet cross sections, which are more exclusive, but still infrared safe.