2. QCD and the parton model

2.1. Gauge invariance and the QCD Lagrangian

Despite the fact that the strong and electromagnetic interactions behave completely different at low energy, the underlying theories are closely related.

Both are gauge theories. In QED, this means that the theory is invariant under a local phase transformation

$$\psi(x) \longrightarrow V(x)\,\psi(x)$$

$$V(x) = e^{i\alpha(x)} \quad \text{"gauge transformation".}$$

The transformations V form the group $U(1)$.

In 1954 Yang and Mills generalized the construction to arbitrary Lie groups. The one relevant for QCD is the group $SU(3)$: unitary 3×3 matrices with determinant 1: $VV^\dagger = 1$, $\det(V) = 1$.

These matrices have $\underset{V_{ab} \in}{2 \cdot 9} - \underset{VV^{+} = \mathbb{1}}{9} - \underset{\det(V) = 1}{1} = 8$ real parameters.

They can be paremeterized in the form

$$V_{ab}(x) = \exp\left[i\, t^{A}_{ab}\, \omega^{A}(x) \right] \qquad \begin{array}{l} A = 1 \dots 8. \\ a, b = 1 \dots 3 \end{array}$$

The matrices t^{A}_{ab} are called the generators of the group SU(3). They fulfill

$$[\, t^{A}, t^{B} \,] = i \underset{\uparrow}{f^{ABC}}\, t^{C}$$

structure constants, totally antisymm.

Note: For SU(2) $t^{A} = \sigma^{A}/2$ Pauli matrices
$A = 1 \dots 3$

$$f^{ABC} = \varepsilon^{ABC}$$

For SU(3) $t^{A} = \dfrac{\lambda^{A}}{2}$ "Gell-Mann matrices"

$$\lambda^{1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \dots .$$

The vector space in which these matrices act is also called color space. $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$ three colors

In order to obtain an invariant Lagrangian, ordinary derivatives had to be replaced by covariant ones

photon field

$$iD_\mu = i\partial_\mu - eA_\mu .$$

In the non-abelian case

Gluon field

$$iD_\mu = i\partial_\mu + g\, A_\mu^A\, t^A$$

or, explicitly,

$$(iD_\mu)_{ab} = i\partial_\mu\, \delta_{ab} + g\, A_\mu^A\, t^A_{ab} .$$

Under a gauge transformation

$$\bar{\psi}(i\not{D} - m)\psi \longrightarrow \bar{\psi}'(i\not{D}' - m)\psi'$$

$$= \bar{\psi} V^+ i\not{D}' V \psi - m\bar{\psi}\psi$$

which is invariant only if

$$iD_\mu' = V(x)\, iD_\mu\, V^+(x)$$

From this equation, we can read off the transformation law of A_μ^a:

inhomogeneous

$$i\partial_\mu + \underline{g A'^A_\mu t^A} = i\partial_\mu + \underline{V(x)[i\partial_\mu V^\dagger(x)]}$$
$$+ \underline{g V(x) A^A_\mu t^A V^\dagger(x)}$$

The field strength tensor is defined as

$$F^A_{\mu\nu} t^A = \frac{1}{ig}[D_\mu, D_\nu]$$
$$= (\partial_\mu A^A_\nu - \partial_\nu A^A_\mu) t^A - ig [A^B_\mu t^B, A^C_\nu t^C]$$
$$= (\partial_\mu A^A_\nu - \partial_\nu A^A_\mu) t^A + g f^{BCA} t^A A^B_\mu A^C_\nu$$

Under a gauge transformation

$$F^A_{\mu\nu} t^A \longrightarrow V(x) F^A_{\mu\nu} t^A V^\dagger(x)$$

Gauge invariant term is

$$\mathrm{tr}[F^A_{\mu\nu} t^A F^{\mu\nu B} t^B] = F^A_{\mu\nu} F^{\mu\nu B}\, \mathrm{tr}[t^A t^B]$$

Can choose t^A, so that $\mathrm{tr}\left[t^A t^B\right] = \frac{1}{2}\delta^{AB}$.

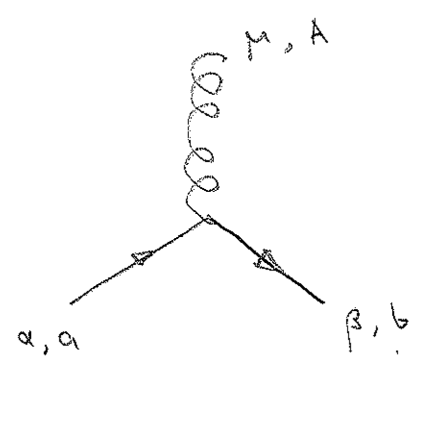
The Lagrangian of QCD then reads

$$\mathcal{L} = -\frac{1}{4} F^A_{\mu\nu} F^{\mu\nu A} + \sum_{i=1}^{n_f} \bar{\psi}_i (i\not{D} - m_i)\psi_i$$

i is the quark flavor index. The standard model includes 6 quark flavors (u, d, c, s, t and b) and each flavor comes in 3 colors. Except for the mass, QCD does not distinguish the different flavors.

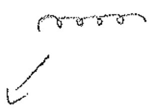
QCD perturbation theory

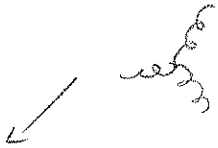
With the Lagrangian at hand, we can now read off the interaction terms and associated Feynman rules.



$$= ig\gamma^\mu_{\beta\alpha} t^A_{ba} \equiv ig\gamma^\mu t^A$$

This is the same as in QED, except for $e \to -g$ and the presence of the color matrix t^A_{ba}.
However, in addition there are now gluon-gluon interactions:



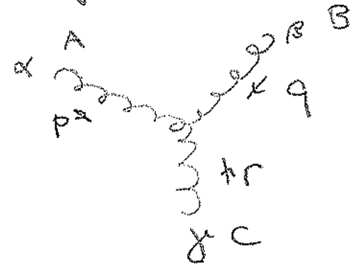


$$-\frac{1}{4} F^{\mu\nu A} F^A_{\mu\nu} = -\frac{1}{4}\left(\partial_\mu A^A_\nu - \partial_\nu A^A_\mu\right)^2$$

$$- g f^{ABC} \left(\partial_\alpha A^A_\beta\right) A^B_\alpha A^C_\beta$$

$$- \frac{g^2}{4} f^{ABE} f^{CDE} A^A_\mu A^B_\nu A^C_\mu A^D_\nu$$

To get the associated Feynman rule, one has to Fourier transform the corresponding terms in the action:



$$\int d^4x \left(\partial_\gamma A^A_\alpha(x)\right) A^B_\alpha(x) A^C_\gamma(x)$$

$$= \int d^4x \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4r}{(2\pi)^4} \frac{p_\gamma}{i} g_{\alpha\beta} \tilde{A}^A_\alpha(p) \tilde{A}^B_\beta(q) \tilde{A}^C_\gamma(r)$$
$$e^{-ix(p+q+r)}$$

$$= \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \int \frac{d^4r}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p+q+r) \frac{p_\gamma}{i} g_{\alpha\beta} \tilde{A}^A_\alpha(p) \tilde{A}^B_\beta(q) \tilde{A}^C_\gamma(r)$$

When this vertex is inserted into a diagram, there are 3! possibilities to contract it:

$$f^{ABC} p_\gamma g_{\alpha\beta} + f^{BAC} q_\gamma g_{\beta\alpha}$$
$$+ f^{CAB} r_\beta g_{\gamma\alpha} + f^{ACB} p_\beta g_{\alpha\gamma}$$
$$+ f^{BCA} q_\alpha g_{\beta\gamma} + f^{CBA} r_\alpha g_{\gamma\beta}$$
$$= f^{ABC} \left[(p-q)_\gamma g_{\alpha\beta} + (r-p)_\beta g_{\gamma\alpha} + (q-r)_\alpha g_{\beta\gamma} \right]$$

Let's check the overall prefactor: The term in $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ contained

$$-g f^{ABC} \frac{p_\beta}{i} g_{\alpha\gamma} = i g f^{ABC} p_\beta g_{\gamma\alpha} .$$

This needs to be multiplied with i (since we consider $iS = i \int d^4x \, \mathcal{L}$), which is consistent with the above expression.

Next, one considers the four-gluon term

$$\Delta \mathcal{L} = -\frac{g^2}{4} f^{ABE} f^{CDE} g_{\alpha\gamma} g_{\beta\delta} A^A_\alpha A^B_\beta A^C_\gamma A^D_\delta$$

Now there are 4! contractions. Four of these are equivalent, which leads to the Feynman rule

$$-ig^2 f^{ABE} f^{CDE} \left(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \right)$$
$$-ig^2 f^{ACE} f^{BDE} \left(g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\delta} g_{\beta\gamma} \right)$$
$$-ig^2 f^{ADE} f^{BCE} \left(g_{\alpha\beta} g_{\gamma\delta} - g_{\alpha\gamma} g_{\beta\delta} \right).$$

The simplest way to quantize QCD is to use the path integral formalism, where one computes probabilities by integrating over all field configurations. In this context, gauge invariance poses a technical difficulty: for each gluon field configuration, there are infinitely many equivalent ones, which are related by gauge transformations.

This by itself is harmless. For example, when we integrate a rotation invariant function over $\mathbb{R}^3$, we'll get $\int d^3r\, f(|\vec{r}|) = \Omega \int dr\, r^2 f(r)$,

where $\Omega = 4\pi$ is the volume of the symmetry group. The problem in our case is that the analogue of Ω is infinite and that it is difficult to separate the integration over the symmetry group from the rest.

A method to solve the problem was found in '67 by Faddeev and Popov. We will skip the derivation and just state the result, which amounts to the following extra terms in the Lagrangian

$$\Delta \mathcal{L} = \underbrace{\frac{1}{2}\lambda}_{\text{arbitrary parameter}} (\partial^\mu A^a_\mu)^2 + \bar{c}^A (-\partial^\mu \underbrace{D^{AB}_\mu}_{\partial_\mu \delta^{AB} + g f^{ABC} A^C_\mu}) c^B$$

arbitrary parameter

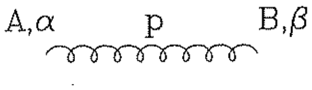
ghost field

anti-commuting scalar field (!)

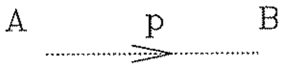
These terms are not gauge invariant, so the theory is no longer manifestly gauge invariant. However, the final results for gauge invariant observables are unchanged by their presence. This implies, in particular, that the result for physical quantities are independent of the gauge fixing parameter.

The ghost fields are unphysical and cancel unphysical contributions from time-like and longitudinally polarized gluon fields. It is possible to use more complicated gauge-fixing procedures, in which the ghost fields are absent but one then ends up with a complicated form of the gluon propagator. (Axial gauges.)

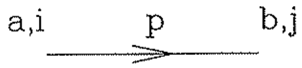
QCD Feynman Rules



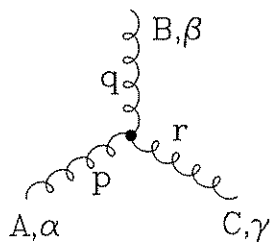
$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\frac{p^{\alpha}p^{\beta}}{p^{2}+i\epsilon}\right]\frac{i}{p^{2}+i\epsilon}$$



$$\delta^{AB}\frac{i}{(p^{2}+i\epsilon)}$$

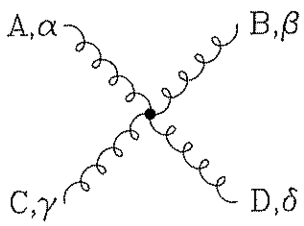


$$\delta^{ab}\frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

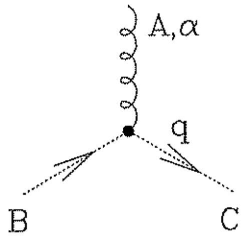


$$+g\ f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

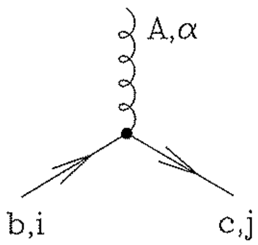
(all momenta incoming, $p+q+r = 0$)



$$\begin{aligned}&-ig^{2}\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]\\&-ig^{2}\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]\\&-ig^{2}\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]\end{aligned}$$



$$-g\ f^{ABC}q^{\alpha}$$



$$+ig\ (t^{A})_{cb}\ (\gamma^{\alpha})_{ji}$$

(from K. Ellis, with $g \to -g$)

2.2. The parton model

Given that hadrons are complicated bound states of quarks and gluons, and given that these constituents were never observed directly, it looks like it is hopeless to use perturbation theory for QCD computations. However it turns out that perturbative computations give reliable results for quantities which are insensitive to low-energy physics. Because of asymptotic freedom, the renormalized coupling $\alpha_s(\mu) = \frac{g^2(\mu)}{4\pi}$ becomes small at large energy scales. During the 60's $e^- p$ scattering results were well described by computations which assumed that the proton was made up of point-like, non-interacting constituents. ("Naive parton model")

Feynman: " We shall ... think of the incoming proton as a box of partons sharing the momentum and practically free."

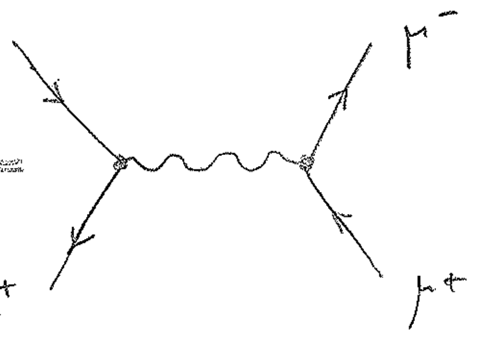
Once asymptotic feedom was discovered things fell into place: the partons could be identified with the quarks and gluons and it became possible to compute perturbative corrections to the predictions of the naive parton model.

To see the corresponding physics in practice, it is simplest to consider $e^+e^- \to$ hadrons and define

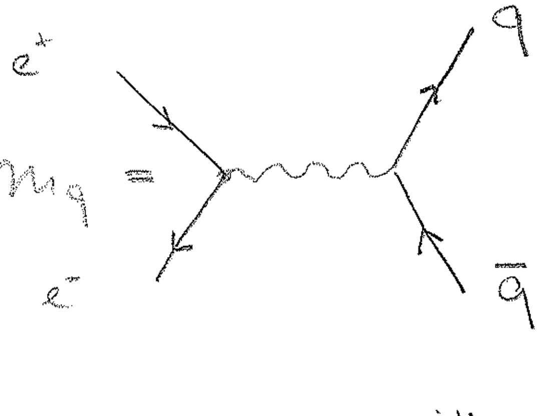
$$R(Q^2) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)},$$

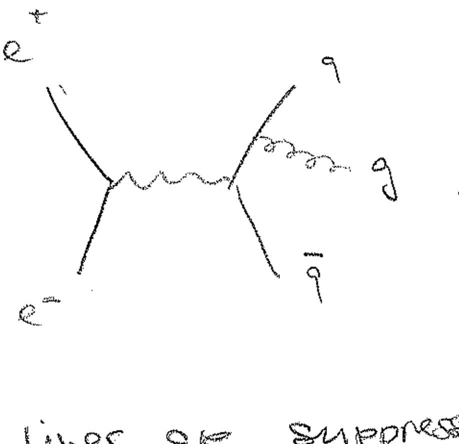
were Q^2 is the cms energy of the collision.

In part 0 of this lecture, we have computed

$\mathcal{M}_\mu =$ 

Let us now be bold and compute $e^+e^- \to$ hadrons in QCD perturbation theory. Since we have no hadrons, we'll simply put quarks and gluons into the final state:

$\mathcal{M}_q =$ 

; 

; ...

The diagrams with gluon lines are suppressed by factors of the coupling g. At lowest order only the first diagram contributes. If $Q^2 \gg m_q$ and $Q^2 \gg m_\mu$, we can neglect the masses.

In this case $\mathcal{M}_q = e_q \mathcal{M}_\mu$

$+2/3$ for u, c, t ; $-\frac{1}{3}$ for d, s, b

$$\longrightarrow \quad \sigma_q = e_q^2 \, \sigma_\mu .$$

To obtain the total hadronic cross section, we sum over all quark flavors which are light enough to be produced.

$$\Longrightarrow \quad R = 3 \cdot \sum_q e_q^2$$

↑ Number of colors.

$$\longrightarrow \quad R = 2 \quad \text{für} \quad Q < 2m_c$$

$$R = \frac{10}{3} \quad \text{für} \quad 2m_c < Q < 2m_b$$

$$R = \frac{11}{3} \quad \text{für} \quad 2m_b < Q \ll m_Z$$

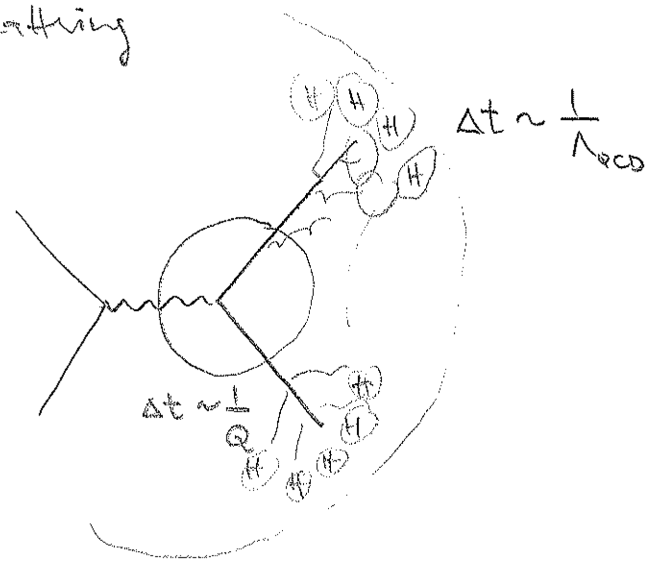
(For $Q \gtrsim m_Z$, we need to include the Z-exchange diagram)

A comparison to experimental measurements is shown in the attached slides. As long as one stays away from thresholds, where the cross section is governed by non-perturbative resonance physics, the above predictions work extremely well.

The physics reason that it works is that we deal with two quite different energy scales:

$Q \gg \Lambda_{QCD} \sim 1\,\text{GeV}$

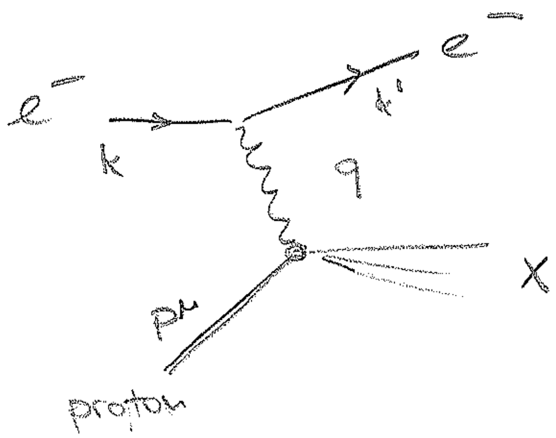
↑ hard scattering ↖ bound-state dynamics.

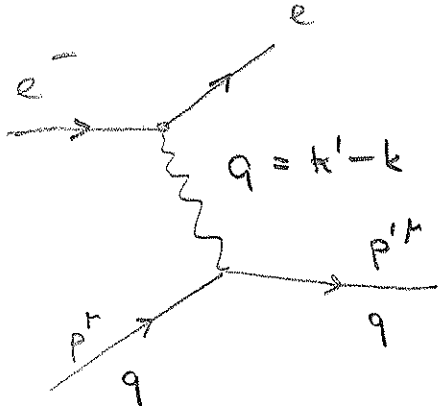


First the quarks are produced and only much later the quarks form hadrons. Since all the quarks and gluons end up in hadrons and since we sum over all final states, the cross section is (at leading power in $1/Q^2$) independent of the complicated hadron dynamics.

There is a rigorous method to derive the result that $\sigma(e^+e^- \to \text{hadrons})$ can be computed perturbatively. The method also allows one to systematically account for Λ^2_{QCD}/Q^2 suppressed power corrections due to hadronisation. This method is the operator product expansion (OPE).

Let us now turn to $e^- p \to e^- + X$, where X denotes an arbitrary hadronic final state. We want to consider the scattering again in the high-energy limit:





We will again go ahead and compute this in perturbation theory.

The leading order diagram is the same as the one for $e^+e^- \to \mu^+\mu^-$, only turned by ninety degrees. The expression for the amplitude is the same, except that we now need particle spinors for all four particles:

$$i\mathcal{M} = \bar{u}(k')\gamma^\mu u(k)\ \bar{u}(p')\gamma_\mu u(p)\ (-ie)^2\ \frac{e_q}{q^2}$$

To get the cross section, we need to square the matrix element. We'll also average over incoming and sum over outgoing spins.

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4 e_q^2}{4q^4} \sum_{\text{spins}} \bar{u}(k')\gamma^\mu u(k)\,\bar{u}(p')\gamma_\mu u(p)$$
$$* \ \bar{u}(k)\gamma^\nu u(k')\,\bar{u}(p)\gamma_\nu u(p')$$

Using $\sum_s u_\alpha(k)\bar{u}_\beta(k) = (\not{k})_{\alpha\beta}$, the matrix element reduces to two Dirac traces:

$$= \frac{e^4 e_q^2}{4q^4}\ \mathrm{tr}\{\not{k}'\gamma^\mu \not{k}\gamma^\nu\}\ \mathrm{tr}\{\not{p}'\gamma_\mu \not{p}\gamma_\nu\}$$

$$= \dots = \frac{2e^4 e_q^2}{\hat{t}^2} \left(\hat{s}^2 + \hat{u}^2 \right)$$

where $\hat{t} = q^2 = (k - k')^2 = -2kk' = -Q^2$

$\hat{s} = (p + k)^2 = 2p \cdot k$

$\hat{u} = (p - k')^2 = -2p \cdot k'$

The hat reminds us that these quantities are partonic, i.e. defined in terms of quark (and gluon) momenta. The partonic cross section is

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{1}{16\pi \hat{s}} \cdot \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{d\hat{\sigma}}{dQ^2}$$

Let us now assume that during the high-energy collision, the proton can be viewed as consisting of quarks, antiquarks and gluons, which move along the proton direction, and carry a fraction ξ of the proton momentum

$$\underbrace{p}_{\text{quark mom.}} = \xi \overbrace{P}^{\text{Proton momentum}} + O(\Lambda_{QCD})$$

$\xi = 0 \dots 1$

The probability to find a quark q with momentum fraction ξ will be given by some nonperturbative function $f_q(\xi)$.

Before computing the cross section, let's introduce kinematical variables:

$$\underline{y = \frac{q \cdot P}{k \cdot P}} = \frac{q \cdot p}{k \cdot p} = \frac{p \cdot k - p \cdot k'}{p \cdot k} = 1 + \frac{\hat{u}}{\hat{s}}$$

$$\rightarrow \hat{u} = (y-1)\,\hat{s}$$

$$x = \frac{Q^2}{2P \cdot q} = \frac{Q^2}{2p \cdot q/\xi} = \frac{\xi Q^2}{2k \cdot p\, y} = \frac{\xi Q^2}{\hat{s}\, y}$$

$$\hat{s} = \frac{Q^2}{x y \xi} \quad ; \quad \hat{s} + \hat{t} + \hat{u} = 0 \Rightarrow \hat{s} y - Q^2 = 0$$

Now let's compute the contibution of quark q to the hadronic cross section:

$$\frac{d\sigma_q}{dQ^2} = \int_0^1 d\xi\, f_q(\xi)\, \frac{d\hat{\sigma}}{dQ^2}$$

Often, one is interested in

$$\frac{d\sigma_q}{dQ^2\,dx} = \int_0^1 d\xi\, f_q(\xi)\, \frac{d\hat\sigma}{dQ^2}\, \delta\Big(x - \underbrace{\frac{Q^2\xi}{\hat s y}}_{=\xi}\Big)$$

$$\frac{d\hat\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left(1 + (y-1)^2\right)$$

Including all quarks and anti-quarks gives

$$\frac{d\sigma}{dQ^2\,dx} = \frac{2\pi\alpha^2}{Q^4} \sum_{i=q,\bar q} e_q^2 f_q(x) \cdot \left(1 + (1-y)^2\right)$$

One can analyze $e^-p \to e^- X$ in general, without making any assumptions on the strong interaction dynamics. In this case, one has:

structure functions

$$\frac{d\sigma}{dQ^2\,dx} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 + (1-y)^2\right) F_1 + \frac{(1-y)}{x}\left(F_2 - 2x F_1\right)\right]$$

where $F_1 = F_1(x, Q^2)$ and $F_2 = F_2(x, Q^2)$.

Our computation gives

$$F_1(x, Q^2) = \sum_{i=q,\bar q} e_q^2 f_q(x) \qquad \text{Q-independent!}$$

$$F_2(x, Q^2) = 2x F_1(x, Q^2) \qquad \text{"Callan-Gross relation"}$$

The fact that the structure functions are (approximately) Q^2-independent is called Bjorken scaling. Higher-order QCD corrections lead to logarithmic violations of the scaling. Also, once gluons come into play, the Callan-Gross relation is violated. Also scalar partons would violate the CG relation. The fact that it holds approximately helped establish that the quarks have spin ½.

By considering also $\nu p \to e X$ and by scattering also on neutrons, one can disentangle f_u, f_d and $f_{\bar{q}}$ and from the violations of Bjorken scaling one obtains f_g. See the associated slides for the result of such an analysis at HERA.