0. From Feynman diagrams to cross sections

In this prelude to the lecture on LHC physics, I will discuss how one computes scattering amplitudes from Feynman diagrams and how one obtains cross sections from the result. I will not derive the Feynman rules – the derivation will be familiar to those who have attended QFT lectures. However, the rules are quite simple and provide an intuitive understanding of QFT, so even if you have not attended a QFT course, you should be able to follow most of the rest of the lecture if once you are familiar with them.

The Feynman rules provide a diagrammatic way of computing amplitudes

Momentum and spin of particle 1; there could be other quantum numbers.

$$\langle q_1 s_1 ; q_2 , s_2 ; \ldots ; q_n ; s_n \text{ out} \mid p_1 r_1 ; p_2 r_2 \text{ in} \rangle$$

$$= (2\pi)^4 \delta(p_1 + p_2 - q_1 - q_2 - \cdots q_n) \, i\mathcal{M} ,$$

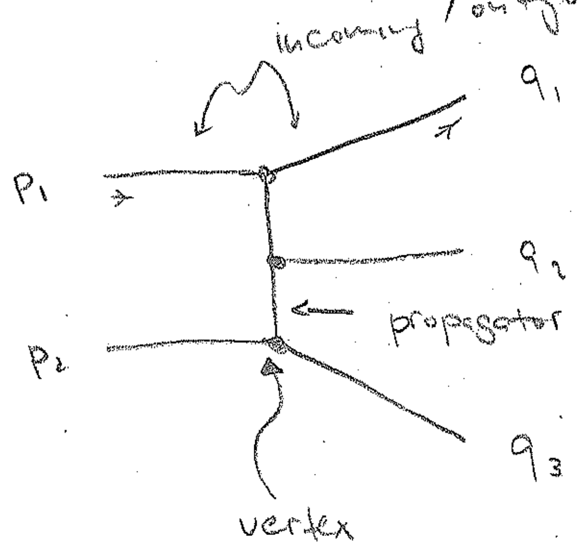
which describes the scattering of the incoming particles with momentum p_1 and p_2 into n outgoing particles.

The probability that the scattering process takes place is proportional to $|\mathcal{M}|^2$.

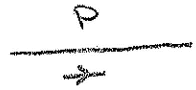
For simplicity, let us start with a scalar field theory with Lagrangian

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 - \frac{g}{3!}\phi^3$$

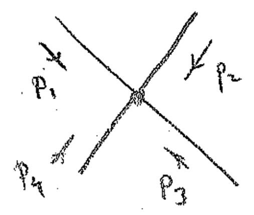
The scattering can be obtained from diagrams such as



The Feynman rules translate such diagrams into mathematical expressions. They are

1.) 

$= \dfrac{i}{p^2 - m^2 + i\varepsilon}$ propagator

2.) 

$= -ig$

Momentum must be conserved at each vertex ! $p_1 + p_2 + p_3 + p_4 = 0$!

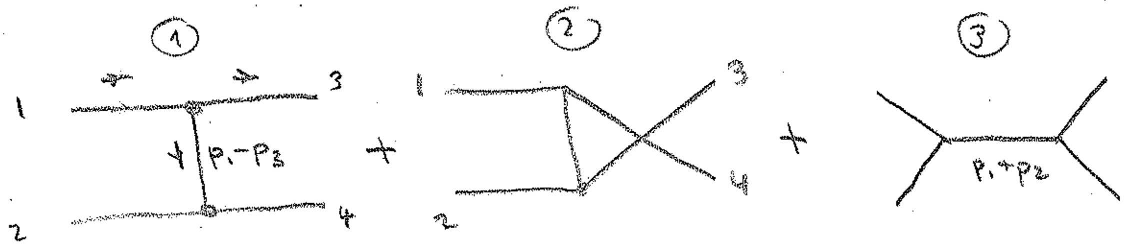
Loops only

3.) Integrate over undetermined momenta $\int \frac{d^4p}{(2\pi)^4}$

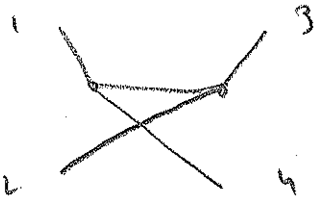
4.) Put symmetry factor

To obtain the scattering amplitude, all possible Feynman diagrams need to be computed.

Example: $p_1 + p_2 \longrightarrow p_3 + p_4$



are possible diagrams. How about



?

This is not a new diagram; by moving the vertices, one finds that it is the same as 2.

The scattering amplitude is

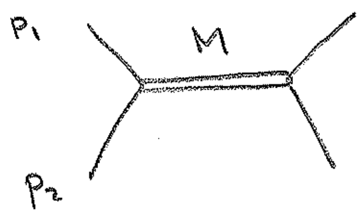
$$i\mathcal{M} = \frac{i}{(p_1-p_3)^2 - m^2}(-ig)^2 + \frac{i}{(p_1-p_4)^2 - m^2}(-ig)^2$$

$$+ \frac{i}{(p_1+p_2)^2 - m^2}(-ig)^2$$

Note that the Amplitude will be large when the denominators are small. For $m = 0$, for example

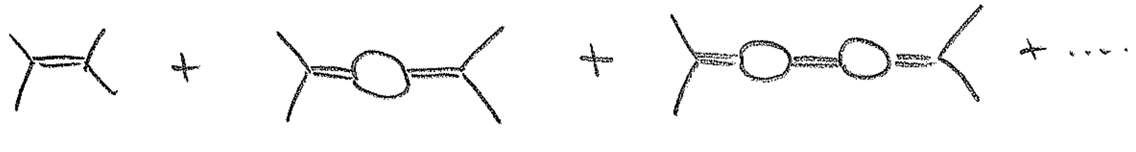
$$\frac{1}{(p_1 - p_3)^2} = \frac{1}{-2 p_1 \cdot p_3} \sim \frac{1}{-E_1 E_3 \theta^2/2}$$ for small scattering angle θ.

Also, if a heavy particle with mass M is produced, a diagram such as



$$\propto \frac{1}{s - M^2} \longrightarrow \infty \quad \text{for} \quad s = (p_1 + p_2)^2 \longrightarrow M^2$$

diverges. In this case, one needs to resum higher order diagrams



this cures the divergence at $s = M^2$. In this region, one can approximate the propagator as $\dfrac{1}{p^2 - m^2 - i \Gamma m}$

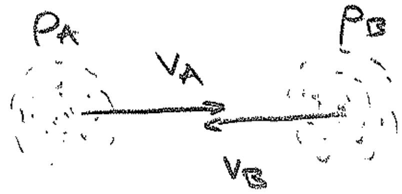
$\uparrow$ width

At colliders, we scatter bunches of particles.

The probability, that scattering occurs is proportional to the density and the relative velocity of the bunches

$$\frac{dP}{dt\, d^3x} = \rho_A(x)\, \rho_B(x)\, |\vec{v}_A - \vec{v}_B| \cdot \sigma$$

Cross section



The cross section σ, on the other hand, is independent of these external factors.

final state phase-space

$$d\sigma = \frac{1}{2E_A\, 2E_B} \frac{1}{|\vec{v}_A - \vec{v}_B|} \prod_{i=1}^{n} \frac{d^3 q_i}{2 q_i^0 (2\pi)^3}$$

$$\cdot \left| \mathcal{M}(p_1 + p_2 \longrightarrow q_1 + q_2 + \ldots + q_n) \right|^2$$

$$\cdot (2\pi)^4\, \delta(p_1 + p_2 - q_1 - q_2 - \ldots - q_n)$$

As an example, consider again $p_1 + p_2 \to p_3 + p_4$ in the scalar ϕ^3 theory. Set $m = 0$ and parametrize: ($E \equiv E_{CMS}$; $\theta \equiv \theta_{CMS}$)

$$p_1^\mu = E\,(1, 0, 0, 1)$$

$$p_2^\mu = E\,(1, 0, 0, -1)$$

$$s = (p_1 + p_2)^2 = 2 p_1 \cdot p_2 = 4E^2$$

$$p_3^\mu = E\,(1, \sin\theta, 0, \cos\theta)$$

$$p_4^\mu = E\,(1, -\sin\theta, 0, -\cos\theta)$$

$$t = (p_1 - p_3)^2 = -2E^2(1 - \cos\theta)$$

$$u = (p_1 - p_4)^2 = -2E^2(1 + \cos\theta)$$

$s + t + u = 0$ ✓

$$V_A = \frac{p_1}{E_1} = 1 \; ; \; V_B = \frac{p_2}{E_2} = -1$$

$$|V_A - V_B| = 2 .$$

$$\sigma = \frac{1}{2s} \int \frac{d^3\vec{q}_1}{(2\pi)^3 2E_1} \int \frac{d^3\vec{q}_2}{(2\pi)^3 2E_2} |\mathcal{M}(E,\theta)|^2$$

$$(2\pi)^4 \delta(2E - E_1 - E_2)\, \delta^3(\vec{q}_1 + \vec{q}_2)$$

$$= \frac{1}{2s\,(2\pi)^2} \int \frac{d^3\vec{q}_1}{(2E)^2} \delta(2E - 2E_1)\, |\mathcal{M}(E,\theta)|^2$$

$(q = |\vec{q}|)$

$$= \frac{1}{2s\,(2\pi)^2} \int_0^\infty dq\, \frac{1}{8}\, \delta(E-q) \int d\varphi \int d\cos\theta\, |\mathcal{M}(E,\theta)|^2$$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{\sin\theta} \frac{d\sigma}{d\theta} = \frac{1}{32\pi s} |\mathcal{M}(E,\theta)|^2$$

Let us now consider QED to see how one deals with particles with spin.

$$\mathcal{L} = \bar{\psi}_\alpha (i\not{D} - m\mathbb{1})_{\alpha\beta}\,\psi_\beta - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$$(\not{V})_{\alpha\beta} = V_\mu (\gamma^\mu)_{\alpha\beta} \qquad \psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

Vector ↑ ↖ Dirac Matrix

$$\{\gamma^\mu, \gamma^\nu\} = \mathbb{1}\; 2g^{\mu\nu}$$

↑ 4×4 matrix

$$iD_\mu = i\partial_\mu - e A_\mu$$

↖ $e = |e|$ electron charge.

$$\bar{\psi}_\beta = \psi^\dagger_\alpha (\gamma^0)_{\alpha\beta}$$

Chiral basis

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

σ^i : Pauli matrices

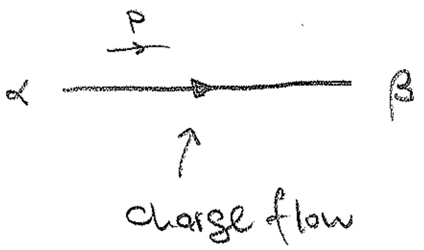
$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Spinors are solutions of the free Dirac equation

$$(\not{p} - m\mathbb{1})_{\alpha\beta}\, u_\beta(p,s) = 0$$

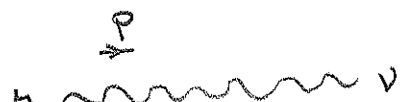
$$(\not{p} + m\mathbb{1})_{\alpha\beta}\, v_\beta(p,s) = 0$$

Feynman Rules



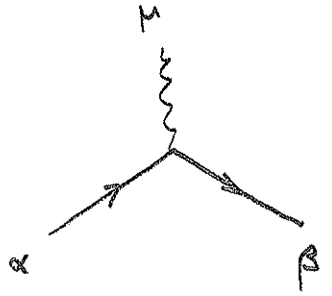
$$= \frac{i}{p^2 - m^2 + i\varepsilon} (\not{p} + m\mathbb{1})_{\beta\alpha}$$

electron propagator



$$= \frac{i}{p^2 + i\varepsilon} (-g_{\mu\nu})$$

photon propagator



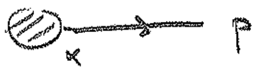
$$= -ie\,\gamma^{\mu}_{\beta\alpha}$$

from $\bar{\Psi}_{\beta}\,\gamma^{\mu}_{\beta\alpha}\,\Psi_{\alpha}\,(-e)\,A_{\mu}$

External lines:

Incoming fermion: 

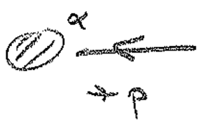
$= \ldots\; u_{\alpha}(p,s)$

Outgoing fermion 

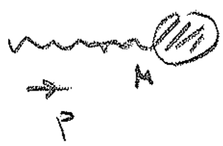
$= \bar{u}_{\alpha}(p,s)\ldots$

Inc. antifermion 

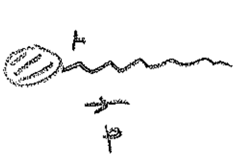
$= \bar{v}_{\alpha}(p,s)\ldots$

Out. antifermion 

$= \ldots\; v_{\alpha}(p,s)$

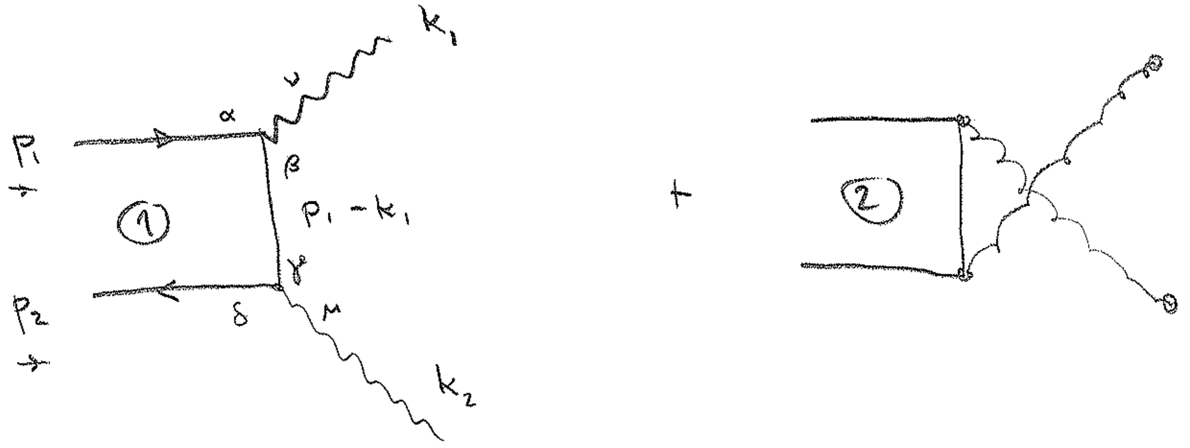
Inc. photon 

$= \varepsilon_{\mu}(p,\lambda)$

Out. photon 

$= \varepsilon^{*}_{\mu}(p,\lambda)$

Example:



$$i\mathcal{M} = \bar{v}_\delta(p_2, s_2)\,(-ie)\,\gamma^\mu_{\delta\gamma}\,\frac{i(\not p_1 - \not k_1 + m\mathbb{1})_{\gamma\beta}}{(p_1-k_1)^2 - m^2}\,\gamma^\nu_{\beta\alpha}(-ie)\,u_\alpha(p_1, s_1)$$

$$\cdot\ \varepsilon_\nu(k_1, \lambda_1)\ \varepsilon_\mu(k_2, \lambda_2) \quad + \quad (2)$$

$$\mathcal{M} = -e^2\,\frac{1}{(p_1-k_1)^2 - m^2}\ \bar{v}(p_2, s_2)\ \not\varepsilon(k_2, \lambda_2)\ (\not p_1 - \not k_1 + m)$$

$$*\ \not\varepsilon(k_1, \lambda_1)\ v(p_1, s_1)$$

$$+ \left(\text{“}\ k_1, \lambda_1 \longleftrightarrow k_2, \lambda_2\ \text{”}\right)$$